- Control limit for $\bar{x}$-Chart

$$CL = \bar{\bar{x}} = 999.32$$

$$UCL = \bar{\bar{x}} + A_2\bar{R} = 999.32 + 0.58\ (4.4) = 1001.872$$

$$LCL = \bar{\bar{x}} - A_2\bar{R} = 999.23 - 0.58\ (4.4) = 996.872$$

Since all the sample points (means) fall within the control limits, therefore the process is under statistical control.

- Control limit for R-Chart

$$CL = \bar{R} = 4.4$$

$$UCL = D_4\bar{R} = 2.115 \times 4.4 = 9.306$$

$$LCL = D_3\bar{R} = 0 \times 4.4 = 0$$

Since all the sample points (ranges) fall within the control limits, the process is under statistical control.

## Conceptual Questions 18A

1. What is statistical quality control? Point out its importance in the industrial world.
2. (a) Distinguish between process control and product control.
   (b) Distinguish between control limits and tolerance limits.
3. What is a control chart? Describe how a control chart is constructed and interpreted.
4. Explain the term 'statistical quality control'. How is process control achieved with the help of control charts? What are the fundamentals underlying the construction of a quality control chart?
5. What do you understand by Statistical Quality Control (SQC)? Discuss briefly its need and utility in industry. Discuss the causes of variations in quality.
6. Explain what are chance causes and assignable causes of variation in the quality of a manufactured product.
7. What do you mean by SQC? What are the advantages when a process is working in a state of statistical control.
8. How does statistical quality control help in industry? Describe the procedure for drawing a control chart during production and indicate how you detect lack of control in the production process.
9. What might cause a process to be out of control?
10. Explain why a process can be out of control even through all samples fall within the upper and lower control limits.
11. What is the purpose of a control chart and what are its features?
12. Is it sufficient to say that a process is stable as long as values of the statistic of interest remain within the control limits? Explain.

## Self-Practice Problems 18A

**18.1** The following data give the weight (in gram) of an automobile part. Five samples of four items each were taken on a random sample basis (at an interval of one hour each). Draw the mean control chart and find out if the production process is in control.

| *Sample Number:* | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| *Observations* | 10 | 10 | 10 | 11 | 12 |
| | 12 | 12 | 10 | 10 | 12 |
| | 10 | 13 | 9 | 9 | 12 |
| | 12 | 13 | 11 | 14 | 12 |

**18.2** A machine is set to deliver packets of a given weight. Ten samples of size 5 each were recorded. Below are given the relevant data:

| *Sample numbers* : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Mean* ($\bar{x}$) : | 15 | 17 | 15 | 18 | 17 | 14 | 18 | 15 | 17 | 16 |
| *Range* (R) : | 7 | 7 | 4 | 9 | 8 | 7 | 12 | 4 | 11 | 5 |

Calculate the values of the central line and the control limits for the mean chart and range chart and then comment on the state of control (conversion factors for $n$ = 5 are $A_2 = 0.58$, $D_3 = 0$, and $D_4 = 2.115$).

[*HP Univ., MCom, 1996*]

**18.3** The overall average of a process you are attempting to monitor is 50 units. The average range is 4 units. What are the upper and lower control limits if you choose to use a sample size of 5?

**18.4** A company produces refrigeration units for food producers and retail food firms. The overall average temperature that these units maintain is 46 degrees Fahrenheit. The average range is 2 degrees Fahrenheit. Samples of six are taken to monitor the process. Determine the upper and lower control chart limits for averages and ranges for these refrigeration units.

**18.5** A machine is set to deliver packets of a given weight. Ten samples of size 5 each were recorded as shown below:

| *Sample number* | *Sample mean* | *Sample average* |
|---|---|---|
| 1 | 12.8 | 2.1 |
| 2 | 13.1 | 3.1 |
| 3 | 13.5 | 3.9 |
| 4 | 12.9 | 2.1 |
| 5 | 13.2 | 1.9 |
| 6 | 14.1 | 3.0 |
| 7 | 12.1 | 2.5 |
| 8 | 15.5 | 2.8 |
| 9 | 13.9 | 2.5 |
| 10 | 14.2 | 2.0 |

Calculate the values for the central line and the control limits for the mean chart and the comment on the state of control (Given $n = 5, A_2 = 0.577, D_3 = 0, D_4 = 2.115$)

## Hints and Answers

**18.1** $\bar{\bar{x}} = 56/5 = 11.2;\ \bar{R} = 12/5 = 2.4$

Control limits: $\bar{\bar{x}} + A_2\bar{R} = 11.2 \pm 0.729\ (2.4);$
$A_2 = 0.729$ for $n = 4$.

**18.2** $\bar{\bar{x}} = \Sigma x/n = 162/10 = 16.2;\ \bar{R} = \Sigma R/n = 74/10 = 7.4$

- $\bar{x}$-chart control limits: $\bar{\bar{x}} \pm A_2\bar{R} = 16.2 \pm 0.58\ (7.4) = 16.2 \pm 4.292;$

  $CL = \bar{\bar{x}} = 16.2$

  All mean values are within UCL and LCL (20.492 to 11.908). The process is in control.

- R-chart control limits:

  $UCL = D_4\bar{R} = 2.115\ (7.4) = 15.651$

  $LCL = D_3\bar{R} = 0(7.4) = 0;\ CL = \bar{R} = 7.4$

  All range values are within UCL and LCL (15.651 and 0). The process is in control.

**18.3** $UCL_{\bar{x}} = 52.308;\quad LCL_{\bar{x}} = 47.692$

$UCL_R = 8.456;\quad LCL_R = 0$

**18.4** $UCL_{\bar{x}} = 46.966;\quad LCL_{\bar{x}} = 45.034$

$UCL_R = 4.008;\ LCL_R = 0$

**18.5** $\bar{\bar{x}} = 135.3/10 = 13.53;\ \bar{R} = 25.9/10 = 2.59$

Control limits for $\bar{x}$-chart

$UCL_{\bar{x}} = \bar{\bar{x}} + A_2R = 13.53 + 0.577\ (2.59) = 15.02$

$LCL_{\bar{x}} = \bar{\bar{x}} - A_2R = 13.53 - 0.577\ (2.59) = 12.04$

Control limits for R-chart

$UCL_R = D_4\bar{R} = 2.115\ (2.59) = 5.48;$

$LCL_R = D_3\bar{R} = 0$

## 18.8 CONTROL CHARTS FOR ATTRIBUTES

Control charts for attributes are used to understand whether products under inspection satisfy or not certain characteristics. In other words, the attribute (quality characteristic) charts are typically based on classification of products or services as defective or non-defective. This class of charts neither include any measurement of variation, nor include anything comparable to a R-chart derived from the range in samples. However, the attribute charts are similar to variable charts in the sense that the control limits are set at three standard errors ($\sigma_p$'s) away from the means of all possible values of the attribute (i.e. defective or non-defective).

### 18.8.1 C-Chart : Control Chart for Defects per Unit

Sometimes the characteristics representing the quality of a product or service are discrete in nature and the data is obtained by counting, such as if machine is idle or working, defects in automobiles, machine components, service rendered by a restaurant or department store or bank, and so on. In such cases, it is more relevant to evaluate performance by keeping track of the number of undesirable occurrences (C), such as number of defects per unit or the number of complaints received, say per 100 customers served or per Rs 10,000 sales.

C-chart is used in situations wherein opportunities for a defect in each production unit or a complaint from a customer are very large while the probability of their occurrence per unit tends to be very small and constant. The outcome of such a sampling process can be described by a Poisson distribution.

The steps for construction of control limits for number of defects where the sample size is constant are follows:

1. If $C_i$ are the number of defects in sample $i$ of size $n$, then the average number of defects per unit (sample) are given by

$$\text{Average defects } \bar{C} = \frac{\text{Number of defects in all samples}}{\text{Total number of samples}}$$

$$= \frac{C_1 + C_2 + \ldots + C_r}{N}$$

$N = 1 + 2 + \ldots + r$

2. Placing control limits using the mean $\bar{C}$ and the standard deviation $\sqrt{\bar{C}}$ of the Poisson distribution as follows:

$$CL = \bar{C}, \quad UCL = \bar{C} + 3\sqrt{\bar{C}}, \text{ and } LCL = \bar{C} - 3\sqrt{\bar{C}}$$

3. The sample points $C_1, C_2, \ldots, C_r$ are plotted as points on a graph paper by taking the sample characteristic C along the $y$-axis and the sample number along the $x$-axis. The control lines are drawn in the same manner as discussed before.
4. With appropriate adjustments, ensure whether a process is under control or not.

**Example 18.4:** During an examination of equal length, the following number of defects were observed: 2, 3, 4, 0, 5, 6, 7, 4, 3, 2. Draw a control chart for the number of defects and comment whether the process is under control or not.

**Solution:** Let C denote the number of defects per piece. Then the average number of defects in 10 samples will be

$$\bar{C} = \frac{\Sigma C}{N} = \frac{2 + 3 + 4 + \ldots + 3 + 2}{10} = \frac{36}{10} = 3.6$$

Hence, control limits are: $\bar{C} = 3.6$

$$UCL = \bar{C} + 3\sqrt{\bar{C}} = 3.6 + \sqrt{3.6} = 3.6 + 5.692 = 9.292$$

$$LCL = \bar{C} - 3\sqrt{\bar{C}} = 3.6 - \sqrt{3.6} = 3.6 - 5.692 = -2.092 \text{ or } 0$$

The control chart for C based on these limits is given in Fig. 18.8.

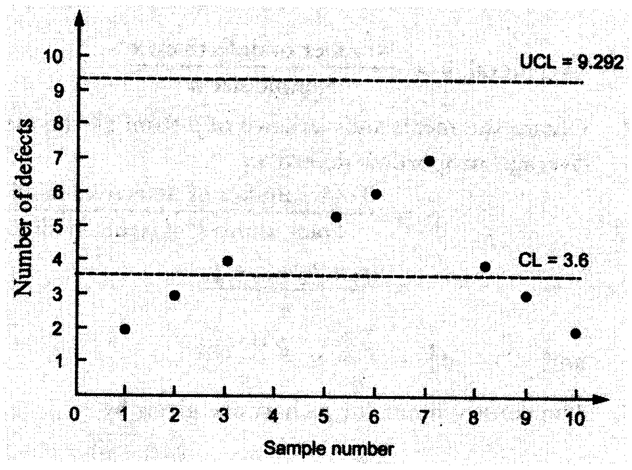



**Figure 18.8**
C-chart

Since all the points are lying within control limits, the process is in control.

**Example 18.5:** Construct a control chart for C, that is, number of defectives, from the following data pertaining to the number of imperfections in 20 pieces of cloth of equal length in a certain make of polyster and infer whether the process is in a state of control: 2, 3, 5, 8, 12, 2, 3, 4, 6, 5, 6, 10, 4, 6, 5, 7, 4, 9, 7, 3.

**Solution:** Let C denote the number of defects per piece. Then

$$\Sigma C = 2 + 3 + 5 + 8 + 12 + 2 + 3 + 4 + 6 + 5 + 6 + 10 + 4 + 6 + 5 + 7 + 4 + 9 + 7 + 3 = 111$$

Control limits: $$\bar{C} = \frac{111}{20} = 5.55$$

$$UCL = \bar{C} + 3\sqrt{\bar{C}} = 5.55 + 3\sqrt{5.55} = 5.55 + 7.08 = 12.63$$

$$LCL = \bar{C} - 3\sqrt{\bar{C}} = 5.55 - 7.08 = -1.53 \text{ or } 0$$

Since none of the points is falling outside the upper and lower control limits, the process is in control.

### 18.8.2 *p*-Chart: Control Chart for Proportion of Defectives

***p*-chart** A control chart used when the output of a process is measured in terms of the proportion defective.

The ***p*-chart** is designed to control the percentage (or proportion) of defectives per sample and is based on the distribution of proportion (or fraction) defectives in each sample. The assumption that attributes that are classified as either good or bad follow the binomial distribution, implies that

(a) there are only two possible outcomes (good or defective),

(b) the outcomes occur randomly, and

(c) the probability of either outcome remains unchanged for each trial.

Since the number of defectives (C) per unit can be converted into a fraction (proportion) defectives by dividing C by the sample size, therefore *p*-chart may be used in place of the C-chart. The *p*-chart has at least two advantages over the C-chart:

1. Expressing the defectives as a percentage (or fraction) of the given population is more meaningful.
2. When sample size varies from sample to sample, the *p*-chart derives more meaningful and simple presentation.

If the sample size is constant, the primary difference in C-chart and *p*-chart chart is only in the computation of the control limits. The steps for construction of control limits for *p*-chart are as follows:

1. Compute the proportion defective items in each sample by dividing the number of defectives $x_i$ recorded in a sample of size $n_i$

$$p_1 = \frac{x_1}{n_1},\quad p_2 = \frac{x_2}{n_2},\quad \ldots,\quad p_i = \frac{x_i}{n_i}$$

In general, $$p = \frac{\text{Number of defectives, } x}{\text{Sample size } n}$$

2. Obtain the mean and variance of $p$ from all the samples combined, i.e.

Average proportion defectives

$$\bar{p} = \frac{\text{Total number of defectives in all the samples combined}}{\text{Total number of items in all the samples combined}}$$

$$= \frac{p_1 + p_2 + \ldots + p_n}{n}$$

and $$\sigma_{\bar{p}}^2 = \frac{\bar{p}\bar{q}}{n} = \frac{\bar{p}(1-\bar{p})}{n}$$

3. The control limits for *p*-chart are given by

$$UCL = \bar{p} + 3\sigma_{\bar{p}} = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

$$LCL = \bar{p} - 3\sigma_{\bar{p}} = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

where $\sigma_{\bar{p}}$ is the standard error (deviation) of proportion. While constructing the *p*-chart it is generally preferred to express results in terms of 'per cent defective' rather than 'fraction defective'. The per cent defective is $100p$. The sampling distribution of $\bar{p}$

can be approximated by a normal distribution whenever the sample size is large with mean $\bar{p}$ and standard deviation $\sigma_{\bar{p}}$.

**Example 18.6:** The following data refer to defects found at the inspection of the first 10 samples of size 100. Use them to obtain the upper and lower control limits for percentage defective in samples of 100. Represent the first ten sample results in the chart you prepare to show the central line and control limits.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Sample number | : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| No. of defectives | : | 2 | 1 | 1 | 3 | 2 | 3 | 4 | 2 | 2 | 0 = 20 |

**Solution:** Since there are 20 defective items in 10 samples each of size 100, therefore

$$\bar{p} = \text{Average proportion defectives} = 20/10 = 2;$$

$$CL = 2$$

$$UCL = p + 3\sigma_{\bar{p}} = \bar{p} + 3\sqrt{\frac{\bar{p}\bar{q}}{n}} = 2+3\sqrt{\frac{2\times 98}{100}} = 2 + 4.2 = 6.2$$

$$LCL = p - 3\sigma_{\bar{p}} = 2 - 3\sqrt{\frac{2\times 98}{100}} = 2 - 4.2 = -2.2 \cong 0$$

The control chart for $p$ based on these limits is given in Fig 18.9

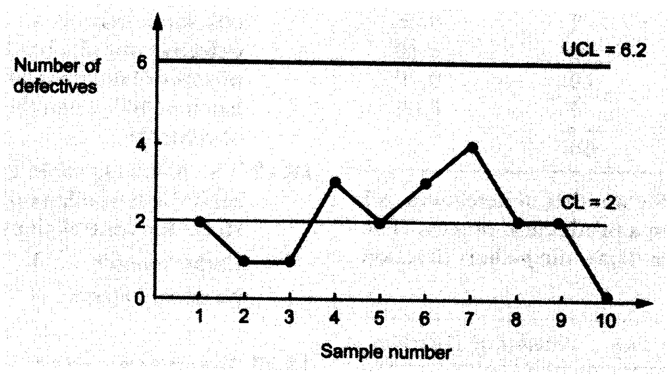



**Figure 18.9**
*p*-chart for Number of Defectives

**Example 18.7:** The average number of defectives in 23 samples of size 2000 rubber belts each, was found to be 16 per cent. Indicate how to construct the relevant control chart.
**Solution:** Given, $n = 23$; Number of rubber belts inspected per sample = 2000, and average fraction defectives per sample, $\bar{p} = 0.16$.

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.16 + 3\sqrt{\frac{0.16\times 0.84}{2000}}$$

$$= 0.16 + 3 \times 0.0082 = 0.1846$$

$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.16 - 3\times 0.0082 = 0.1354$$

$$CL = \bar{p} = 0.16$$

### 18.8.3 *np*-Chart : Control Chart for Total Number of Defectives

If the sample size is constant, then ***np*-chart** is constructed to control the actual number of defectives per sample. The construction and interpretation of a *np*-chart is similar to that of the *p*-chart because in this case we can directly plot the 'number' rather than the fraction or percentage of defectives. In *np*-chart, the central control limit is drawn at *np* instead of *p*. The upper and lower control limits are given by:

***np*-chart:** A control chart used to monitor the output of a process in terms of the number of defective items.

$$UCL = np + 3\sqrt{npq} \text{ and } LCL = np - 3\sqrt{npq}$$

If $p$ is unknown, it can be estimated by sample values. The best estimate of $p$ is $\bar{p}$. Then control limits are written as:

$$UCL = n\bar{p} + 3\sqrt{n\bar{p}\bar{q}} \text{ and } LCL = n\bar{p} - 3\sqrt{n\bar{p}\bar{q}}$$

where, $n\bar{p}$ = average number of defectives per sample of constant size, $n$ from the process.

## Self-Practice Problems 18B

**18.6** If the average fraction defective of a large sample of size 2000 products is 0.1537, calculate the control limits. What modifications do you need if the sample size is variable?

**18.7** The following table gives the inspection data relating to 10 samples of 100 items each, concerning the production of bottle corks. Contruct a $p$-chart.

| *Sample Number* | *Size of Sample* | *Number of Defectives* | *Fraction Defective* |
|---|---|---|---|
| 1 | 100 | 5 | 0.05 |
| 2 | 100 | 3 | 0.03 |
| 3 | 100 | 3 | 0.03 |
| 4 | 100 | 6 | 0.06 |
| 5 | 100 | 5 | 0.05 |
| 6 | 100 | 6 | 0.06 |
| 7 | 100 | 8 | 0.08 |
| 8 | 100 | 10 | 0.10 |
| 9 | 100 | 10 | 0.10 |
| 10 | 100 | 4 | 0.04 |
| | 1000 | 60 | |

**18.8** The following data gives the number of defectives in 5 independent samples from a production process. The samples are of varying size. Draw the $p$-chart (fraction defective chart).

| *Sample Number* | *Sample Size* | *Number of Defectives* |
|---|---|---|
| 1 | 2000 | 400 |
| 2 | 1000 | 150 |
| 3 | 1000 | 120 |
| 4 | 600 | 80 |
| 5 | 400 | 50 |
| | 5000 | 800 |

**18.9** An inspection of 10 samples of size 400 each from 10 lots revealed the following number of defective units:

17, 15, 14, 26, 9, 4, 19, 12, 9, 15

Calculate control limits for the number of defective units. State whether the process is under control or not.

**18.10** From the information given below, construct an appropriate control chart.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *Sample number (each of 100)* : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| *Number of defective* : | 12 | 7 | 9 | 8 | 10 | 6 | 7 | 11 | 8 |

State your conclusions. Write all the steps in the construction of the above chart including formulae for both upper and lower control limits.

**18.11** The past records of a factory using quality control methods show that on an average, 4 items produced are defective out of a batch of 100. What is the maximum number of defective items likely to be encountered in a batch of 400, when the production process is in a state of control?

**18.12** The following table gives the number of defects observed in 8 woollen carpets passing as satisfactory. Construct the control chart for the number of defects.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Carpet number* : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| *Number of defects* : | 3 | 4 | 5 | 6 | 3 | 3 | 5 | 3 | 6 | 2 |

[*Raj Univ., BCom, 1998*]

**18.13** Twenty tape-recorders were examined for quality control test. The number of defects for each tape-recorder are given below:

2, 4, 3, 1, 1, 2, 5, 3, 6, 7, 3, 1, 4, 2, 3, 1, 6, 4, 1, 1

Prepare a C-chart. What conclusions do you draw from it [*GND Univ., MBA, 1997*]

## Hints and Answers

**18.6** $\bar{p} = 0.1537$, $\bar{q} = 1 - 0.1537 = 0.8463$ ; $n = 2000$

Control limits: $\bar{p} \pm 3\sqrt{\dfrac{\bar{p}\bar{q}}{n}}$

$$= 0.1537 \pm 3\sqrt{\frac{0.1537 \times 0.8463}{2000}}$$

$$= 0.1537 \pm 0.0241$$

$$\text{CL} = \bar{p} = 0.1537$$

**18.7** $\bar{p} = 0.06$, $\bar{q} = 0.94$, $n = 100$

Control limits: $\bar{p} \pm 3\sqrt{\dfrac{\bar{p}\bar{q}}{n}}$

$$= 0.06 \pm 3\sqrt{\frac{0.06 \times 0.94}{100}}$$

$$= 0.06 \pm 3\,(0.0237) = 0.06 \pm 0.0711$$

and $\quad \text{CL} = \bar{p} = 0.06$

**18.8** $\bar{p} = \dfrac{\Sigma \text{ defectives}}{\Sigma n} = \dfrac{800}{5000} = 0.16$, $\bar{q} = 0.84$.

Control limits: $\bar{p} \pm 3\sqrt{\dfrac{\bar{p}\bar{q}}{n}} = \bar{p} \pm 3\sqrt{\dfrac{0.16 \times 0.84}{n}}$

$= \bar{p} \pm \dfrac{1.1}{\sqrt{n}}$ where $n$ is variable.

Calculations for $3\sigma$ limits are shown below:

| *Samples* | *Sample Size* | *Defectives* | *Fraction Defectives* | $\sqrt{\frac{pq}{n}} = \frac{1.1}{\sqrt{n}}$ | *UCL* $\bar{p} + 1.1/\sqrt{n}$ | *LCL* $\bar{p} - 1.1/\sqrt{n}$ |
|---|---|---|---|---|---|---|
| 1 | 2000 | 400 | 0.20 | $1.1/\sqrt{2000} = 0.0246$ | 0.16+0.0246 = 0.1846 | 0.16 – 0.0246 = 0.1354 |
| 2 | 1000 | 150 | 0.15 | $1.1/\sqrt{1000} = 0.0348$ | 0.16+0.0348 = 0.1948 | 0.16 – 0.0348 = 0.1252 |
| 3 | 1000 | 120 | 0.12 | $1.1/\sqrt{1000} = 0.0348$ | 0.16+0.0348 = 0.1948 | 0.16 – 0.0348 = 0.1252 |
| 4 | 600 | 80 | 0.13 | $1.1/\sqrt{600} = 0.0449$ | 0.16+0.0449 = 0.2049 | 0.16 – 0.0449 = 0.1151 |
| 5 | 400 | 50 | 0.125 | $1.1/\sqrt{400} = 0.0550$ | 0.16+0.0550 = 0.2150 | 0.16 – 0.055 = 0.1050 |

**18.9** $\bar{p} = \frac{\Sigma \text{ defectives}}{10 \times 400} = \frac{140}{4000} = 0.035,\ \bar{q} = 0.965$

Control limits: $n\bar{p} \pm 3\sqrt{n\bar{p}\bar{q}}$

$= 400 \times 0.035 \pm 3\sqrt{400 \times 0.035 \times 0.965}$

$= 14 \pm 3\sqrt{13.5} = 14 \pm 11.0267$

CL $= n\bar{p} = 14.$

**18.10** $\bar{p} = \frac{\Sigma \text{ defectives}}{9 \times 100} = \frac{79}{900} = 0.087$ ;

$\bar{q} = 1 - \bar{p} = 0.913$

Control limits: $n\bar{p} \pm 3\sqrt{n\bar{p}\bar{q}}$

$= 100 \times 0.087 \pm 3\sqrt{100 \times 0.087 \times 0.913}$

$= 8.7 \pm 3\sqrt{7.9431} = 8.7 \pm 8.45$

CL $= n\bar{p} = 8.7$

**18.11** $n = 4 \times 100 = 400,\ \bar{p} = 4/100 = 0.04,$

$\bar{q} = 1 - \bar{p} = 0.96$

Control limits: $n\bar{p} \pm 3\sqrt{n\bar{p}\bar{q}}$

$= 400 \times 0.04 \pm 3\sqrt{400 \times 0.04 \times 0.96}$

$= 16 \pm 3\sqrt{15.36} = 16 \pm 11.7576$

CL $= n\bar{p} = 16$

**18.12** $\bar{C} = \frac{\Sigma C}{n} = \frac{40}{10} = 4$

UCL$= \bar{C} + 3\sqrt{\bar{C}} = 4 + 3\sqrt{4} = 10$;

LCL $= \bar{C} - 3\sqrt{\bar{C}} = 4 - 6 = -2 \cong 0$

**18.13** $\bar{C} = \frac{\Sigma C}{n} = \frac{60}{20} = 3$

UCL $= \bar{C} + 3\sqrt{\bar{C}} = 3 + 3\sqrt{3} = 8.19$;

LCL $= \bar{C} - 3\sqrt{\bar{C}} = 3 - 3\sqrt{3} = -2.19 \cong 0$

## 18.9 SAMPLING PLAN FOR ATTRIBUTES AND VARIABLES

When it is not physically possible or economically desirable to exercise direct control on a process, we have the option of acceptance sampling to cletermine whether the lot or batch of goods (or items) should be rejected or accepted.

### 18.9.1 Acceptance Sampling

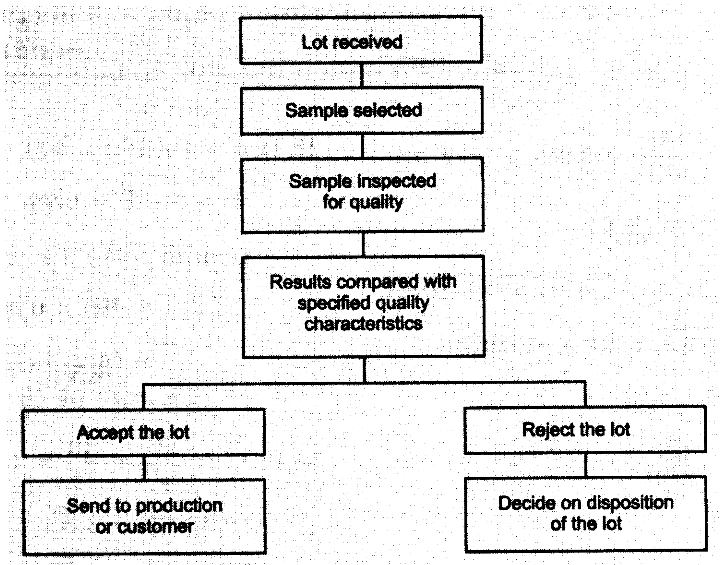
Acceptance sampling involves the inspection of a random sample from a batch (or lot) of raw material, purchased part, or finished product for measuring their quality against predetermined standards. The entire lot is either rejected or accepted on the basis of the quality of goods or items in the sample. Rejected lots may either be returned to the supplier or be inspected 100 per cent at the producer's expense, followed by replacement of defective units by good items.

Since the judgment is based on a sample, there is always a risk of making an error of accepting a bad lot or rejecting a good lot. The general steps of acceptance sampling are shown in Fig. 18.10.

**Advantages of acceptance sampling** A few advantages of acceptance sampling are:

1. Minimizes the total expected cost resulting from sampling errors.
2. The only possible procedure when testing is destructive.
3. Less product damage due to less handling and testing.
4. Acceptability of the incoming products and outgoing products is more compared to 100 per cent inspection where inspection is monotonous.
5. Corrective action may be taken for an ongoing process as and when required.
6. This presumes an agreement between producer and consumer as to what constitutes 'good' and 'bad' quality and the acceptable risk of error for each quality level.
7. Less manpower is required for quality inspection.

**Figure 18.10**
Steps of Acceptance Sampling Procedure

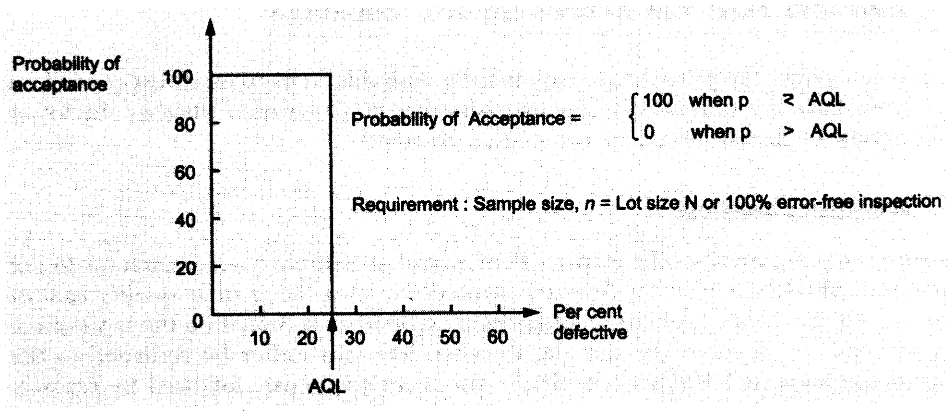



## 18.10 SPECIFYING AN ACCEPTANCE SAMPLING PLAN

To specify an acceptance sampling plan, let us first define the following terms:

**Acceptable Quality Level (AQL)** This is the minimum level of quality acceptable in a given lot. It is expressed as a decimal or percentage defectives in a lot that can be considered satisfactory by the consumer. For example, if acceptable quality is 20 per cent defectives in a lot of 1000 items, then the AQL is 20/100 = 2 per cent.

**Figure 18.11**
Acceptable Quality Level



### 18.10.1 Types of Acceptance Sampling Plans

The following three types of acceptance sampling plans are commonly used:

**Single Sampling Plan** When the decision whether to accept a lot or reject it is made on the basis of only one sample, the acceptance plan is called a *single sampling plan*. This is the simplest type of sampling plan. In any systematic plan for single sampling, three things are specified:

**Acceptance sampling:** A statistical procedure in which the number of defective items found in a sample is used to determine whether a lot should be accepted or rejected.

(i) Number of items N in the lot from which the sample is to be drawn.
(ii) Size of sample $n$ drawn from the lot of N items.
(iii) The acceptance number $c$.

The action plan of a single sample is shown in Fig. 18.12 and the summary of the decision rule for the single sample acceptance sampling plan is as follows:

- Accept the lot if $d \leq c$
- Reject lot if $d > c$

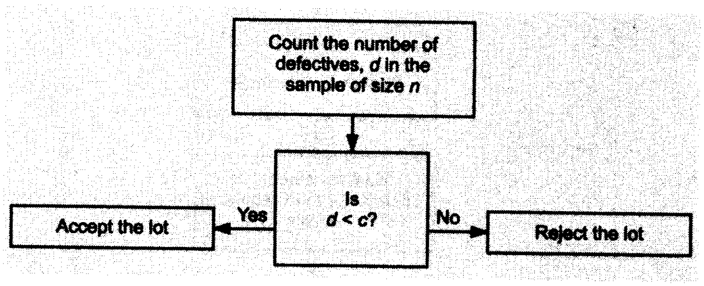



**Figure 18.12**
Single Sampling Plan

**Double Sampling Plan** In the single sampling plan, the decision with regard to acceptance or rejection of a lot is based on only single sample from the lot. However, double sampling involves the possibility of putting off the decision on the lot until a second sample is drawn from the lot. A lot may be accepted at once if the first sample is good enough or rejected at once if the first sample is bad. If the first sample is neither good enough nor bad enough, the decision is based on the results of the first and second samples combined. In a double sampling plan, the following four parameters are specified:

$n$ = size of the sample
$c_1$ = acceptance number for the first sample (the maximum number of defectives or rejects allowed in the first sample)
$r_1$ = a prespecified number of defectives (rejects), $r_1 > c_1$
$c_2$ = acceptance number for the two samples combined (the maximum number of defectives allowed in the two samples.

The logic of a double sampling plan is shown in Fig. 18.13 and the summary of the decision rules for the double acceptance sampling plan is as follows:

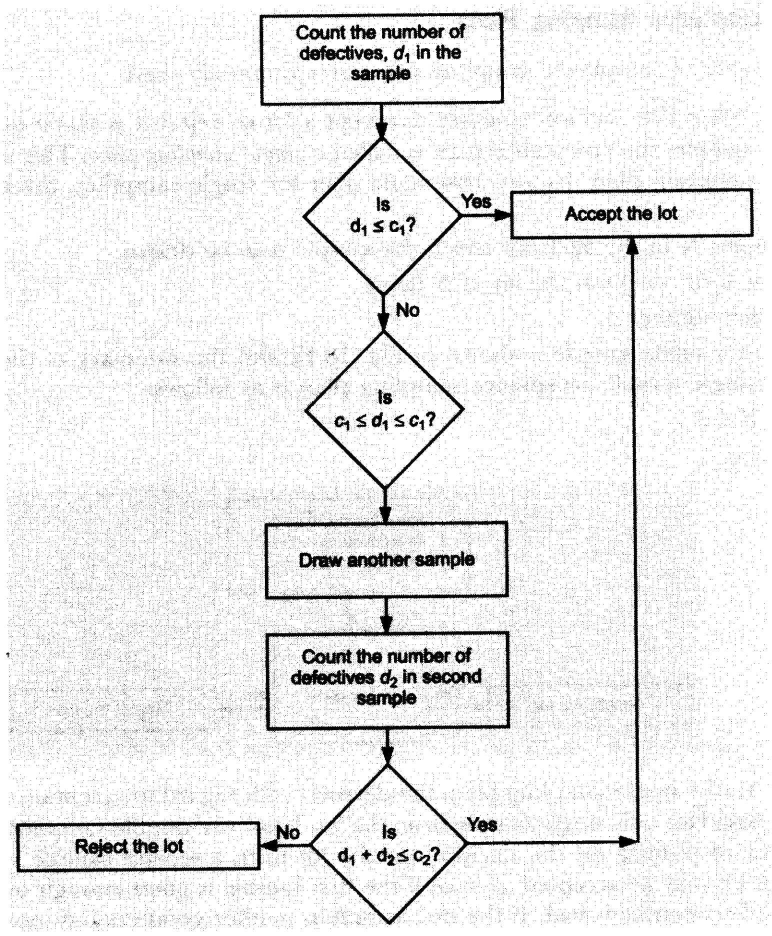
- First sample:
  - Accept the lot if $d_1 \leq c_1$
  - Reject the lot if $d_1 > c_1$
- Take second sample if $c_1 < d_1 < r_1$
- Second sample:
  - Accept the lot if $d_1 + d_2 \leq c_2$
  - Reject the lot if $d_1 + d_2 > c_2$

*Advantages of Double Sampling Plan*

A double sampling plan has two possible advantages over a single sampling plan:

(i) It may reduce the total cost of inspection. Consequently in all cases in which a lot is accepted or rejected on the first sample, there may be considerable saving in total inspection cost. It is also possible to reject a lot without completely inspecting the entire second sample.
(ii) A double sampling plan has the psychological advantage of giving a second chance to inspect the second lot of items because to some people, especially the producer, it may seem unfair to reject a lot on the basis of a single sample.

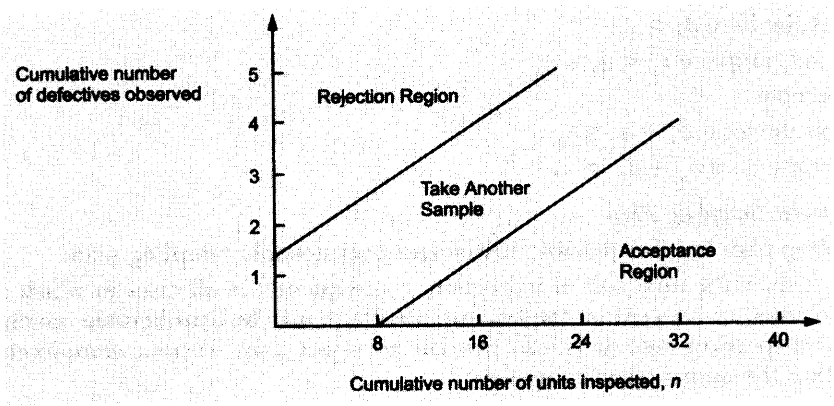
Figure 18.13
Double Sampling Plan




**Multiple (or Sequential) Sampling Plan** Just as a single sampling plan may defer the decision on acceptance or rejection until a second sample has been taken, other plans may permit to draw few more samples in sequential order before a decision is reached. Plans permitting three or more samples to be drawn are referred to as multiple or sequential sampling plan.

In a multiple sampling plan, a decision must be reached within the maximum number of allowed samples as specified in advance. However, in a sequential sampling plan, the sampling may continue until the cumulative evidence is conclusive. In either case, a sample may consist of one or several units from the submitted lot. After each sample is taken, the cumulative evidence for all defectives observed leads to a decision to either (i) accept the lot, (ii) reject the lot, or (iii) take another sample, as shown in Fig. 18.14.

Figure 18.14
Multiple Sampling Plan



**Example 18.8:** A company is producing an item whose weight is normally distributed with standard deviation $\sigma$ = 8 gm. Shipments averaging less than 200 gm are considerd poor quality and the company would like to minimize such shipments. Design a sampling plan for a sample of size $n$ = 25 that will limit the risk of rejecting lots that average 200 gm to 5 per cent.

**Solution:** In this problem it is assumed that the distribution of sample means is approximately normal with mean $\mu$ = 200 and standard error:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{8}{\sqrt{25}} = 1.6 \text{ gm.}$$

The limit for permissible defective level $c$ with 95 per cent confidence is given by

$$c = \mu - z\sigma_{\bar{x}} = 200 - 1.64\ (1.6) = 197.4 \text{ gm}$$

where $z$ = 1.64, value corresponding to area under normal curve.

Hence take a random sample of $n$ = 25 and determine the mean weight. If $\bar{x}$ > 197.4 gm, then accept the shipment. Otherwise reject it.

**Example 18.9:** In Example 18.8, design a sample plan so that

(a) The probability of rejecting a lot with an average weight of 200 gm is 0.05.

(b) The probability of accepting a lot with an average weight of 196 gm is 0.10.

**Solution:** Let us first find simultaneous equations defining the reject limit $c$ in terms of $z$. Then solve these equations for $n$ and substitute it back into either one of the equation to find $c$.

The two equations for $c$ are:

(a) From above: $$c = \mu_1 - z\frac{\sigma}{\sqrt{n}} = 200 - 1.64\frac{8}{\sqrt{n}}$$

(b) From below: $$c = \mu_2 + z\frac{\sigma}{\sqrt{n}} = 196 + 1.28\frac{8}{\sqrt{n}}$$

Equating the two equations for $c$, we get

$$200 - \frac{1.64 \times 8}{\sqrt{n}} = 196 + \frac{1.28 \times 8}{\sqrt{n}} \quad \text{or} \quad n = \left(\frac{23.40}{4}\right)^2 = 34$$

Substituting value of $n$ in the first equation, we have

$$c = 200 - \frac{1.64 \times 8}{\sqrt{34}} = 197.7 \text{ gm}$$

Hence, take a random sample of $n$ = 34 and determine the mean weight. If $\bar{x}$ > 197.7 gm then accept the shipment otherwise reject it.

## 18.11 DETERMINING ERROR AND OC CURVE

The **operating characteristics** (OC) **curve** describes how well an acceptance plan discriminates between good and bad lots. A curve pertain to a specific plan—a combination of $n$ (sample size) and $c$ (acceptance number or maximum number of defectives that will permit the acceptance). The OC curve is intended to show the probability that the given plan will accept lots of various (unknown) quality levels. In other words, the OC curve shows the percentage of lots that would be accepted if a large number of lots of any stated quality are inspected.

**Operating characteristic curve:** A graph showing the probability of accepting the lot in terms of percentage defective in the lot. It helps to determine whether a particular acceptance sampling plan meets both the producer's and the consumer's risk requirements.

Construction of an OC curve requires that the decision-maker specify in advance that what producer and consumer agree to be 'good' and 'bad' quality and what risk each side will accept as a result of sampling error. This information shall help in determining the sample size $n$ and acceptance number $c$ of a sampling plan that can be applied to incoming lots to distinguish between good and bad lots with the agreed risk.

For every set of $n$ (sample size) and $c$ (acceptance level) values, a family of curves can be drawn. For any particular set of $n$ and $c$ values, OC curves show as to how well the sampling plan is able to distinguish between good and bad lots. A typical OC curve is shown in Fig. 18.15 based on the data shown in Table 18.3 for selected binomial probabilities for samples of size $n$ = 15 and $c$ = 0.

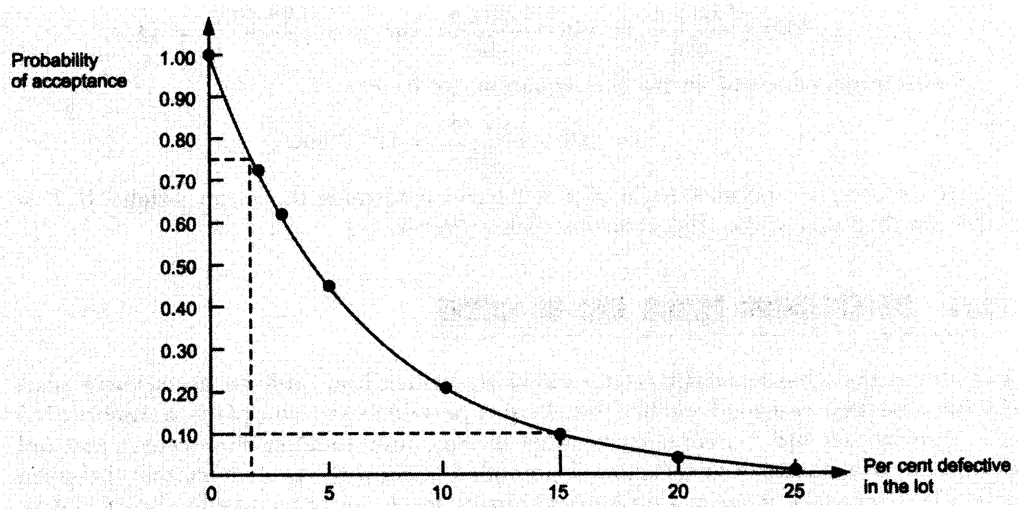
**Table 18.3: Binomial Probabilities of Accepting the Lot for Samples of Size $n$ = 15 and $c$ = 0**

| *Probability of Defective in a Lot* | *Probability of Accepting the Lot* |
|---|---|
| 0.01 | 0.8601 |
| 0.02 | 0.7386 |
| 0.03 | 0.6333 |
| 0.04 | 0.5421 |
| 0.05 | 0.4633 |
| 0.10 | 0.2059 |
| 0.15 | 0.0874 |
| 0.20 | 0.0352 |
| 0.25 | 0.0134 |

From Fig. 18.15, it can be seen that for 2 per cent defectives, the probability of acceptance is about 75 per cent. Similarly, if the percentage of defectives in the lots were 10 per cent, then the probability of defectives is 15 per cent. In other words, if per cent defective acceptable is increased from 2 per cent to 15 per cent, then probability of accepting a lot is only 10 per cent.

In general, Poisson distribution can also be used to determine the probability of $c$ or less defectives. It is useful where $p < 0.10$ or $np < 5$ and lot size is at least 10 times the sample size. For example, for a lot that contains 5 per cent defectives ($p = 0.05$) and a sample size 100 ($n = 100$), that is, $np = 5$, the probability of 2 or less defectives is approximately 0.12. If the lot fraction defective is 1 per cent ($p = 0.01$) and sample size is 100 ($n = 100$) that is, $np = 1$, the probability of 2 or less defective is approximately 0.92. These results give two plots on the OC curve for the acceptance sampling plan where the sample size is 100 and the acceptance level $c = 2$. Other points may be calculated in the same way.

**Figure 18.15**
Operating Characteristic (OC) Curve for $n$ = 15 and $c$ = 0




### 18.11.1 Producer and Consumer Risk

Since under a sampling plan a decision is made as to whether to accept or reject a lot on the basis of a sample, there is a possibility of (a) rejecting a lot which was actually acceptable according to the quality standard, this is termed as *producer's risk*, and (b) accepting of poor quality by the buyer, this risk is called the *buyer's risk*. Rejecting a satisfactory lot (also called Type I error) creates a risk for the producer of the lot due to unwarranted inspection and replacement costs. Thus a large sample is required to minimize the producer's risk. However, accepting poor quality lots (also called Type II error) creates a risk for the consumer of the lot because he bears the cost. Table 18.4 combines the concepts of Type I and Type II error as discussed in chapter on hypothesis testing

**Table 18.4: Producer and Consumer Errors**

| *Decision* | *State of the Lot* | |
|---|---|---|
| | *$H_0$ True* *Good-Quality Lot* | *$H_0$ False* *Poor-Quality Lot* |
| *Accept the Lot* | Correct decision | Type II Error (Accepting a poor-quality lot) |
| *Reject the Lot* | Type I Error (Rejecting a good-quality lot) | Correct decision |

**Producer's risk ($\alpha$):** This is the probability of rejecting a good lot, that is, the lot where percentage defective is AQL or less. Producers intend to keep this risk low because they usually have the responsibility of replacing all defective items in the rejected lot or of paying for a new lot to shipped to the consumer. If a good lot is rejected, it is referred to as a Type I sampling error.

**Producer's risk:** The risk of rejecting a good-quality lot. This is of Type I error.

**Lot tolerance per cent defective (LTPD):** This is the quality level of a lot that is considered bad. It is expressed as percentage defectives in a lot that is considered as the most unsatisfactory or bad quality the consumer can tolerate. For example, if an unacceptable quality level is 70 defects in a lot of 1000, then the LTPD is 70/1000 = 7 per cent defective.

**Consumer's risk ($\beta$):** This is the probability of accepting a lot of unacceptable quality LTPD. The consumer wants to keep this risk low. If a bad lot is accepted, it is referred to as a Type II sampling error.

**Consumer's risk:** The risk of accepting a poor-quality lot. This is of Type II error.

Figure 18.16 shows an ideal OC curve, in which if the lot contains percentage defectives equal to the AQL or less (i.e., 1 or 2 per cent), then the probability of acceptance by the consumer is 1.0, that is, no chance of rejecting the lot. Similarly, if the lot contains percentage defectives more than the AQL, then the probability of acceptance by the consumer is zero, that is, there is no chance that it will be accepted.

In general, as the percentage defectives ($p$) in a lot increases, the probability of acceptance decreases. However, the probability of acceptance depends upon the selection of the sample size $n$ and the acceptance number $c$.

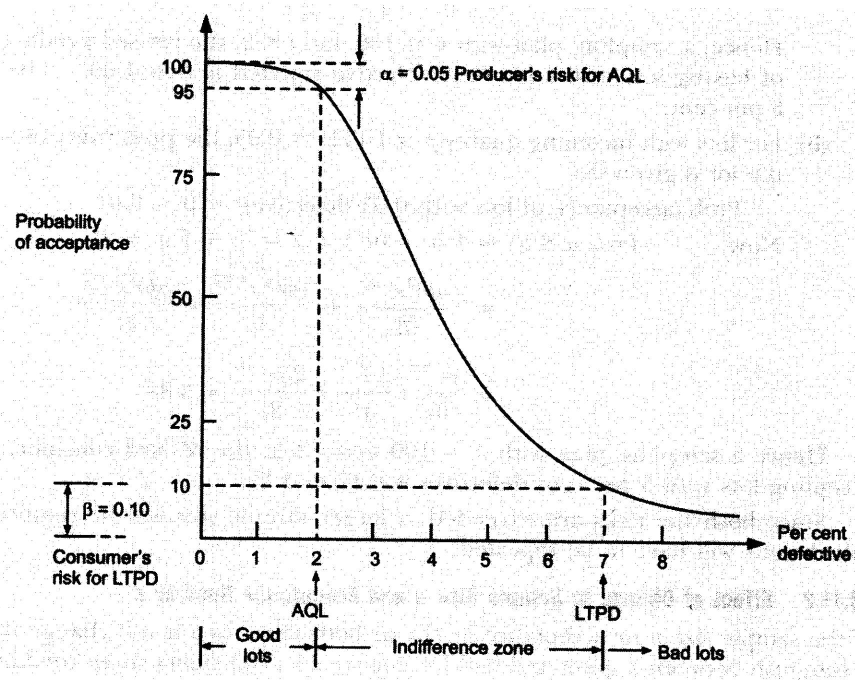



**Figure 18.16**
An OC Curve Showing Producer's and Consumer's Risk

The theoretical relationship between $\alpha$, $\beta$, AQL, and LTPD is shown in Fig. 18.16. The $\alpha$ risk at the AQL level and the $\beta$ risk at the LTPD level establish two points from which the sample size $n$ and acceptance level $c$ are determined. Given these two points, the OC curve can then be drawn to describe the risk characteristics of the specific sampling plan. However, standard tables and charts are available which help reduce the amount of work required to find a set of $n$ and $c$ values that meet the limits set by the $\alpha$, $\beta$, AQL, and LTPD values.

**Example 18.10:** A shipment of 1000 items is to be inspected on a sampling basis. The producer and consumer have agreed to adopt a plan as shown below:

| *Quality level* | *Risk* |
|---|---|
| AQL = 0.01 | $\alpha = 0.05$ |
| LTPD = 0.05 | $\beta = 0.10$ |

Construct an OC curve of the sampling plan $n = 100$, $c \le 2$, and indicate whether this plan satisfies the requirement.

**Solution:** Since the shipment is accepted when there are less than or equal to 2 defectives in the sample, therefore we wish to know the probability $P(c \le 2)$, given the alternative values of the population. We can simplify the calculation by approximating the binomial with Poisson distribution with a mean equal to $\lambda = np$.

$$P(c = r) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r} = \frac{(np)^r e^{-np}}{r!}, \text{ for } np < 5, n > 20, p < 0.10$$

The values chosen for the sample size $n$ and acceptance number $c$ must satisfy the following relationship:

(a) For lots with incoming quality : $p$ = AQL = 0.01 we can find the probability of acceptance of the lot as:

Prob (acceptance with 0.01 defectives) = $1 - \alpha = 1 - 0.05 = 0.95$

Now Prob ($c \le 2$) = $P(c = 0) + P(c = 1) + P(c = 2)$

$$= \frac{(np)^0 e^{-np}}{0!} + \frac{(np)^1 e^{-np}}{1!} + \frac{(np)^2 e^{-np}}{2!}$$

$$= \frac{e^{-1}}{0!} + \frac{e^{-1}}{1!} + \frac{e^{-2}}{2!} = 0.92$$

Hence, a sampling plan with $n = 100$ and $c \le 2$, the revised producer's risk ($\alpha$) of having lots with 1 per cent defective rejected is now $1.00 - 0.92 = 0.08$ or 8 per cent.

(b) For lots with incoming quality $p$ = LTPD = 0.05, the probability of accepting of the lot is given by:

Prob (acceptance of lots with 0.05 defectives) = $\beta = 0.01$

Now Prob ($c \le 2$) = $P(c = 0) + P(c = 1) + P(c = 2)$

$$= \frac{(np)^0 e^{-np}}{0!} + \frac{(np)^1 e^{-np}}{1!} + \frac{(np)^2 e^{-np}}{2!}$$

$$= \frac{e^{-5}}{0!} + \frac{5e^{-5}}{1!} + \frac{25e^{-5}}{2!} = 0.88$$

Hence a sampling plan with $n = 100$ and $c \le 2$, the revised consumer risk ($\beta$) of accepting lots with 5 per cent defectives is 0.12 or 12%.

Since both the risks are exceeded, a larger sample size will be required and the calculations will have to be repeated.

### 18.11.2 Effect of Change in Sample Size *n* and Acceptance Number *c*

If the sample size $n$ or acceptance level $c$ or both vary, then it will change the ability to distinguish between a good and bad lot. Figure 18.17(a) shows three combinations of $n$ and $c$ values. It may be noted from this figure that as the value of $n$ increases, the value of $c$ also increases.

On the other hand, if the sample size $n$ is held constant and acceptance number varies, then OC will shift away from the origin, but the shape and slope remain relatively the same. Such a change will increase the probability of rejection rate for all quality levels and therefore there is need to have 100 per cent inspection. This is shown in Fig. 18.17(b).

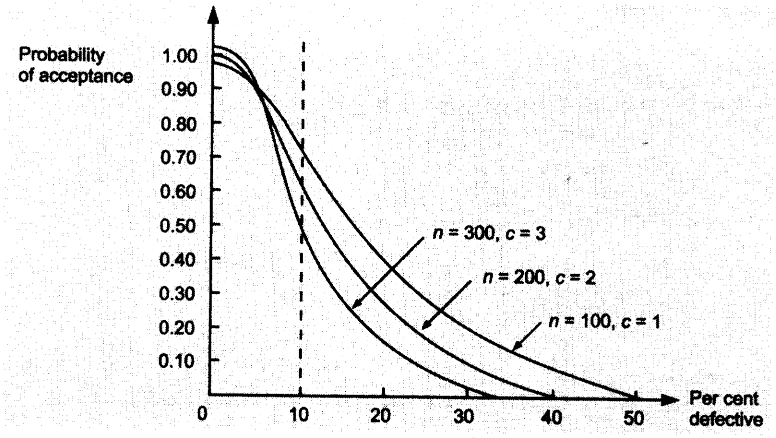



**Figure 18.17 (a)**
OC Curve for Variable $n$ and $c$

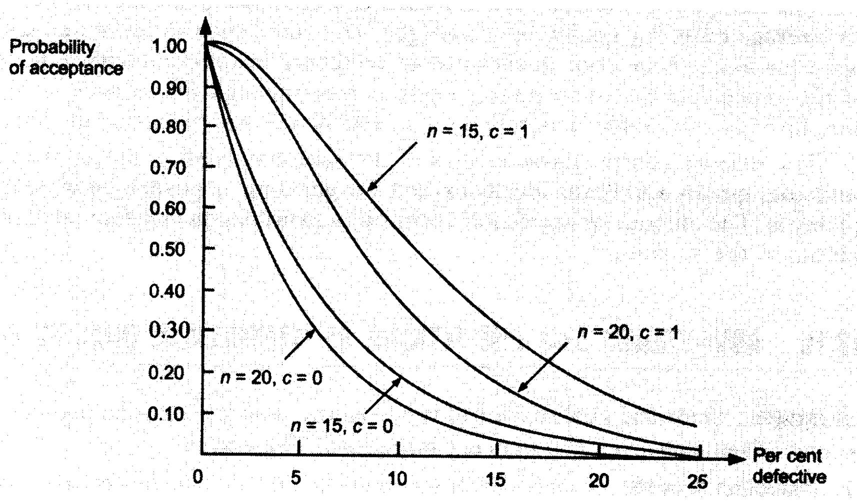



**Figure 18.17 (b)**
OC Curve for Fixed $c$ and variable $n$

### 18.11.3 Average Outgoing Quality (AOQ)

If a rejected lot is subjected to 100 per cent inspection, then the acceptance sampling plan gives definite assurance that the average outgoing quality will not exceed certain limits. The random sample of size '$n$' is inspected and any defective found in the sample are replaced by a good part so that sample ends up with only good parts. Based on the number of defectives, $c'$ found in the sample, the entire lot is accepted if $c' \leq c$ and rejected if $c' > c$. If the lot is rejected, it is subjected to 100 per cent inspection and all defectives found are replaced by good parts. In this case the entire lot of N parts is free of defectives. If the lot is accepted, then there is the risk that some defective parts have been passed. The formula for average outgoing quality for an accepted lot can be stated as follows:

$$\text{AOQ} = \frac{\text{Average number of defective} \times 100}{\text{Lot size}} = \frac{P_d \times P_a (N - n)}{N}$$

where $P_d$ = percentage of defectives in a lot

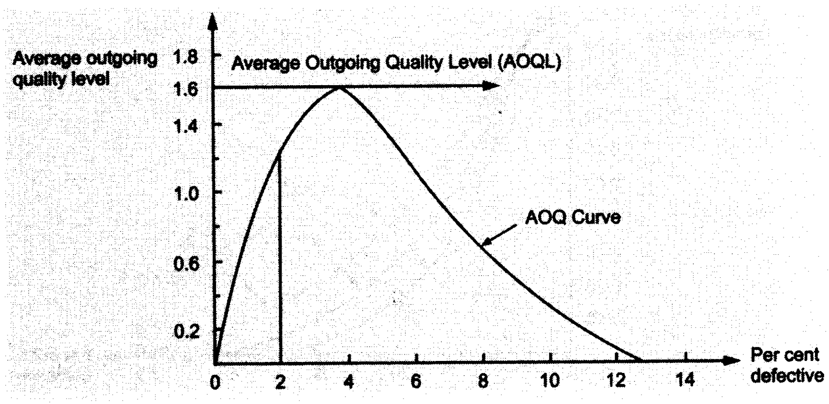
$P_a$ = probability of accepting a lot with $P_d$ defectives

N = lot size

$n$ = sample size

From this relationship we can develop an OC curve for any acceptance sampling plan showing the AOQ for different levels of incoming quality (defective). Such a curve can be plotted by assuming different values of actual incoming quality. These figures can then be substituted in the formula to compute the AOQ. Each calculation for different incoming quality levels determines a point on the AOQ curve as indicated in Fig. 18.18. The AOQ curve in Fig 18.18 is based on a sampling plan with $n = 50$, $c = 1$, and $N = 1000$.

**Figure 18.18**
AOQ Curve for $n = 50$, $c = 1$, Lot Size $N = 1000$




The AOQ curve starts at zero defective and increases to a maximum level, known as the average outgoing quality limit (AOQL). The curve then decreases approaching zero again for lots of very poor quality due to rectifying inspection of the rejected lots. The AOQL represents the worst quality likely to pass through the inspection provided by a sampling plan. Its value depends on $n$, $c$, and N and not on actual incoming quality $p$.

The essence of the characteristics of the sampling plan is simply that the average outgoing quality will never ideally exceed 1.6 per cent, regardless of what the incoming quality is. The amount of inspection required to maintain quality standards automatically adjusts to the situation.

## 18.12 ADVANTAGES AND LIMITATIONS OF STATISTICAL QUALITY CONTROL

**Advantages** Statistical quality control is one of the tools of scientific management. It has several advantages over 100 per cent inspection. These are:

1. **Reduction in costs:** Since only a fraction of the output is inspected, costs of inspection are greatly reduced.
2. **Greater efficiency:** Not only is there reduction in costs but the efficiency also goes up because much of the boredom is avoided, the work of inspection being considerably reduced.
3. **Easy to apply:** An excellent feature of quality control is that it is easy to apply. Once the system is established, it can be operated even by a person who has not had extensive specialized training or a highly mathematical background. It may appear difficult only because the statistical principles on which it is based are unrecognized or unknown. However, as these principles are actually based on commonsense, the quality control method finds wide application.
4. **Early detection of faults:** Quality control ensures an early detection of faults and hence a minimum waste of rejected production. The moment a sample point falls outside the control limits it is taken to be a danger signal and necessary corrective action is taken. On the other hand, with 100 per cent inspection, unwanted variations in quality may be detected at a stage when a large amount of faulty products have already been produced. Thus there would be a big wastage. A control chart, on the other hand, provides a graphic picture of how the production is proceeding and tells management where to look for trouble.

5. **Adherence to specifications:** Quality control enables a process to be brought into and held in a state of statistical control, that is, a state in which variability is the result of chance causes alone. So long as a statistical control continues, specifications can be accurately predicted in the future, which even 100 per cent inspection cannot guarantee. Consequently, it is possible to assess whether production processes are capable of turning out products which will comply with the given set of specifications.
6. **The only course:** In certain situations 100 per cent inspection cannot be carried out without destroying all the products inspected: for example, testing breaking strength of chalks, proofing of ammunition, and so on. In such cases if 100 per cent inspection methods are followed, then all the items inspected will be spoiled. In such a case sampling must be resorted to and the application of SQC techniques ensures not only that the quality is controlled but also that valid inferences about the total output are drawn from the samples.
7. **To determine the effect of changed process:** With the help of control charts one can easily detect whether or not a change in the production process results in a significant change in quality.
8. **Statistical quality control ensures overall co-ordination:** Statistical quality control provides a basis upon which the difference arising among the various interests in an organization can be resolved. In some instances, for example, production engineers may set specifications that are so 'tight' that the operating staff cannot meet them economically and consequently there is an unnecessarily high scrapping rate. In other instances, the specifications may be too loose, and product quality will be sacrificed unnecessarily. In either type of case, the control record provides a valuable aid in solving the problem of getting the operating and engineering forces together on the basis of a common understanding. Information on plant capabilities and customer requirements must also be considered in relation to the quality control limits and records of performance and, finally, it should be possible to determine the best practical balance between the cost of quality and the sales value of a product.

   The following diagram gives a summary of the advantages of quality control.

**Limitations** Despite several advantages of statistical quality control, it is believed that it is not a treatment for all quality evils. The techniques of quality control should not be used mechanically. Instead these should be matched to the process being studied. The application of standard procedures without adequate study of the process is extremely dangerous, and has in the past led to statistical methods being discredited. Statistical methods applied on a production process are only an information service, and as such must be conditioned by the process to which they are applied. Unless they are used as a part of general quality awareness they may only lead to a false sense of security. The responsibility of quality and process decisions rests with the manager in charge of the process and not with the statistician. The charts do not reduce the manager's responsibility.

**Quality Control—Approach and Benefits**

**Planned collection of data, analysis, and interpretation**

↓

- *Improvement in quality*
- *Reduction in cost per unit*
- *Reduction in scrap*
- *Saving in excess use of materials*
- *Removing production bottlenecks*
- *Reduction in inspection*
- *Evaluation of scientific tolerances*
- *Maintenance of operating efficiency*
- *Quality consciousness*
- *Greater customer satisfaction*

↓

**Enhanced Productivity**

## Conceptual Questions 18B

13. Discuss the basic principles underlying control charts. Explain in brief the construction and use of $p$-chart and C-chart.
14. What is a control chart? Explain in brief the construction and use of mean chart, $p$-chart, and range chart.
15. (a) What is acceptance sampling? Point out the role of an operating characteristic curve (OC curve).
    (b) Critically examine the different types of acceptance sampling plans.
16. (a) Discuss briefly the need and utility of statistical quality control in industry. Also point out its limitations, if any.
    (b) What are the various types of control charts known to you? Explain them with examples.
17. 'Quality control is attained most efficiently, of course, not by the inspection operation itself but by getting at the causes'. Comment on the statement. Describe the various devices employed for the maintenance of quality in a uniform flow of manufactured products.
18. Describe control charts for $\bar{x}$ and R, derive expressions for their control limits. What are the advantages of the $\bar{x}$-chart over the R-chart?
19. Explain the term 'Statistical quality control'. How is process control achieved with the help of control charts? What are the fundamentals underlying the construction of quality control charts?
20. What do you understand by statistical quality control (SQC)? Discuss briefly its need and utility in industry. Discuss the causes of variations in quality.
21. Explain what are 'chance causes' and 'assignable causes' of variation in the quality of manufactured products.
22. What do you mean by SQC? What are the advantages when a process is working in a state of statistical control?
23. How does statistical quality control help in industry? Describe the procedure for drawing a control chart during production and indicate how you detect lack of control in the production process.
24. What do the terms 'producer's risk' and 'consumer's risk' mean?

## Formulae Used

1. Mean of overall sample means

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \ldots \bar{x}_n}{n}$$

2. Average range

$$\bar{R} = \frac{R_1 + R_2 + \ldots + R_n}{n}$$

3. Control limits for an $\bar{x}$-chart: Process mean and standard deviation are unknown

$$\bar{\bar{x}} \pm A_2 \bar{R}$$

$$UCL = \mu + 3\sigma_{\bar{x}}$$

$$LCL = \mu + 3\sigma_{\bar{x}}, \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

4. Control Limits for an R-Chart

$$UCL = \bar{R} D_4; \quad LCL = \bar{R} D_3$$

5. Control limits for a $p$-chart

$$UCL = p + 3\sigma_{\bar{x}};$$

$$LCL = p - 3\sigma_{\bar{x}}, \quad \sigma_{\bar{x}} = \sqrt{\frac{p(1-p)}{n}}$$

6. Control limits for an $np$-chart

$$UCL = np + 3\sqrt{np(1-p)}$$

$$LCL = np - 3\sqrt{np(1-p)}$$

7. Binomial probability function for acceptance sampling

$$P(x = r) = \frac{n!}{r!(n-r)!} p^r q^{n-r}, \; q = 1 - p$$

## Chapter Concepts Quiz

### True or False

1. The producer's risk is the probability that a lot will be rejected despite the quality level meeting the acceptable quality level. (T/F)
2. If a 95.5 per cent level of confidence is desired, the $\bar{R}$-chart limits will be set at $\pm 2\sigma$. (T/F)
3. The $\bar{x}$-chart and R-chart are constructed to resolve assignable variations in a process. (T/F)
4. Attribute inspection measures the values of the dimensions of inspected parts. (T/F)
5. Variable inspection is used to determine good parts from defectives. (T/F)

6. Quality cannot be inspected into a product. (T/F)
7. The quality variations resulting from bad raw materials are called chance variations. (T/F)
8. The inevitable variation in product quality results from chance causes. (T/F)
9. The measure of the performance of an acceptance sampling plan is the OC curve. (T/F)
10. Multiple sampling permits more than two samples to reach a decision regarding the quality of the lot .(T/F)
11. The OC curve displays the discriminatory power of a sampling plan. (T/F)
12. The OC curve is useful for comparing the performance of various acceptance plans. (T/F)
13. Benchmarking is one of the five basic concepts of TQM. (T/F)
14. Statistical process control is one of the tools of TQM. (T/F)
15. The OC curve cannot be used when product quality varies in an admissible range. (T/F)

## Multiple Choice

16. The basic issues relating to inspection include
    (a) how much and how often to inspect
    (b) where to inspect
    (c) when to inspect
    (d) all of these
17. The type of inspection that classifies items as being either good or defective is
    (a) variable inspection (b) attribute inspection
    (c) fixed inspection (d) all of these
18. The type of chart used to control the number of defects per unit of output is
    (a) $\bar{x}$-chart (b) R-chart
    (c) $p$-chart (d) none of these
19. Control charts for attributes are
    (a) $p$-charts (b) $\bar{C}$-charts
    (c) R-charts (d) $\bar{x}$-charts
20. C-charts are based on the
    (a) Poisson distribution (b) normal distribution
    (c) Erlang distribution (d) binomial distribution
21. If a sample of parts are measured and the mean of the sample measurement is outside the tolerance limits,
    (a) the process is out of control and the cause can be established
    (b) the process is in control, but not capable of producing within the established control limits
    (c) the process is within the established control limits with only natural causes of variation
    (d) all of these
22. If a sample of parts is measured and the mean of the sample measurement is in the middle of the tolerance limits but some parts measure too low and other parts measure too high,
    (a) the process is out of control and the cause can be established
    (b) the process is in control, but not capable of producing within the established control limits
    (c) the process is within the established control limits with only natural causes of variation
    (d) all of these
23. Acceptance sampling
    (a) involves taking random samples (or batches) of incoming raw materials and measuring them against predetermined standards
    (b) is more economical than 100 per cent inspection
    (c) may be of either a variable or attribute type although attribute inspection is more common in the business environment
    (d) all of these
24. A measure of the performance of an acceptance sampling plan is
    (a) producer's risk (b) consumer's risk
    (c) OC curve (d) none of these
25. Process control is achieved through
    (a) control charts
    (b) acceptance sampling plans
    (c) both (a) and (b)
    (d) none of these
26. The product control is achieved through
    (a) acceptance sampling plans
    (b) control charts
    (c) both (a) and (b)
    (d) none of these
27. The statistical techniques used in statistical quality control are:
    (a) control charts
    (b) acceptance sampling plans
    (c) both (a) and (b)
    (d) none of these
28. Variations due to assignable causes are due to
    (a) faulty process
    (b) operator's mistake
    (c) poor quality of raw material
    (d) all the these
29. The faults due to assignable causes
    (a) can be removed
    (b) cannot be removed
    (c) can sometimes be removed
    (d) all the these
30. $\bar{x}$ and R-charts are
    (a) charts for attributes
    (b) charts for variables
    (c) charts for number of defects
    (d) none of these

## Concepts Quiz Answers

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1. T | 2. T | 3. T | 4. F | 5. F | 6. T | 7. F | 8. T | 9. T |
| 10. T | 11. T | 12. T | 13. T | 14. T | 15. F | 16. (d) | 17. (b) | 18. (d) |
| 19. (a) | 20. (a) | 21. (a) | 22. (b) | 23. (d) | 24. (a) | 25. ( ) | 26. (a) | 27. (c) |
| 28. (d) | 29. (a) | 30. (b) | | | | | | |

# Review Self-Practice Problems

**18.14** Measurement of averages and ranges (R) from 20 samples each of size 5 give the following results: $\bar{\bar{x}} = 99.6$, $\bar{R} = 7.0$. Determine the values of the control limits for drawing a mean chart.

**18.15** What is meant by Statistical Quality Control? State clearly the theoretical assumptions behind the control chart technique.

The following data shows the values of sample mean $\bar{x}$ and the Range R for ten samples of size 5 each. Calculate the values for the central line and control limits for mean chart and ranges chart, and determine whether the process is in control.

| Sample : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Mean : | 11.2 | 11.8 | 10.8 | 11.6 | 11.0 | 9.6 | 10.4 | 9.6 | 10.6 | 10.0 |
| Range : | 7 | 4 | 8 | 5 | 7 | 4 | 8 | 4 | 7 | 9 |

(conversion factors for $n = 5$ are $A_2 = 0.577$, $D_3 = 0$, $D_4 = 2.115$) *[M.C. Kaktiya Univ., MCom, 1999]*

**18.16** You are given the value of sample means ($\bar{x}$) and the ranges (R) for ten samples of size 5 each. Draw mean and range charts and comment on the state of control of the process:

| Sample : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Mean : | 43 | 49 | 47 | 44 | 45 | 37 | 51 | 46 | 43 | 47 |
| Range : | 5 | 6 | 5 | 7 | 7 | 4 | 8 | 6 | 4 | 6 |

You may use the following control chart constants: For $n = 4$, $A_2 = 0.58$, $D_3 = 0$, and $D_4 = 2.115$.

*[Delhi Univ., MCom, 1999]*

**18.17** Draw a suitable control chart for the following data predating to the number of foreign-cultured threads (considered as defects) in 15 pieces of cloth of 2 m × 2 m in a certain make of synthetic fibre and state your conclusions.

7, 12, 3, 20, 21, 5, 4, 3, 10, 8, 0, 9, 6, 7, 20

**18.18** Construct a control chart for the mean and range of the following data on the basis of fuses, sample of 5 being taken every hour (each set of 5 has been arranged in ascending order magnitude).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 42 | 42 | 19 | 36 | 42 | 51 | 60 | 18 | 15 | 69 | 61 | 61 |
| 65 | 45 | 24 | 54 | 51 | 74 | 60 | 20 | 30 | 109 | 90 | 78 |
| 75 | 68 | 80 | 89 | 57 | 75 | 72 | 27 | 39 | 113 | 93 | 94 |
| 78 | 72 | 81 | 77 | 59 | 78 | 95 | 42 | 62 | 118 | 109 | 109 |
| 87 | 90 | 81 | 84 | 78 | 132 | 138 | 60 | 84 | 453 | 112 | 136 |

Comment on whether the production seems to be under control. *[Gujarat Univ., MBA, 1983]*

**18.19** A company manufactures paper clips and other stationary products. Although inexpensive paper clips have provided the firm with a high margin of profitability, the percentage defective for paper clips produced by the company has been averaging 2.5 per cent. Samples of 200 clips are taken. Establish the upper and lower control limits for this process at 99.7 per cent confidence.

**18.20** Following are the number of defects noted in the final inspection of 30 bales of woollen cloth:

0, 3, 1, 4, 2, 2, 1, 3, 5, 0, 2, 0, 0, 1, 2, 4, 3, 0, 0, 0, 1, 2, 4, 5, 0, 9, 4, 10, 3, and 6

Compute the values for an appropriate control chart and give your comments.

*[Kurukshetra Univ., MCom, 1996]*

**18.21** The number of defects in 20 items are given below:

| Item : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Defects : | 2 | 0 | 4 | 1 | 0 | 8 | 0 | 1 | 2 | 0 |
| Item : | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| Defects : | 6 | 0 | 2 | 1 | 0 | 3 | 2 | 1 | 0 | 2 |

Prepare a suitable control chart and draw your conclusions. *[Pune Univ., MBA, 1997]*

**18.22** Samples of 100 tubes are drawn randomly from the output of a process that produces several thousand units daily. Samples are inspected for quality and defective tubes are counted. The results of 15 samples are shown below:

| Sample : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Defective tubes : | 8 | 10 | 13 | 9 | 8 | 10 | 14 | 6 |
| Sample : | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| Defective tubes : | 10 | 13 | 18 | 15 | 12 | 14 | 9 | |

On the basis of the information given above, prepare a control chart for fraction defectives. What conclusion do you draw from the control chart?

*[Ranchi Univ., MBA, 1996]*

## Hints and Answers

**18.15** $UCL_{\bar{x}} = 14.295$, $LCL_{\bar{x}} = 7.025$; $CL_{\bar{x}} = 10.55$
$UCL_R = 13.32$, $LCL_R = 0$, $CL_R = 6.3$

**18.16** $UCL_{\bar{x}} = \bar{x} + A_2\bar{R} = 44.2 + 0.58 \times 5.8 = 47.567$

$LCL_{\bar{x}} = \bar{x} - A_2\bar{R} = 44.2 - 0.58 \times 5.8$

$= 40.836$; $CL_{\bar{x}} = \bar{x} = 44.2$;

$UCL_R = D_4\bar{R} = 2.115 \times 5.8 = 123$;

$LCL_R = D_3\bar{R} = 0$; $CL_R = \bar{R}$

**18.17** $\bar{C} = \dfrac{\Sigma C}{n} = \dfrac{135}{15} = 9$;

$UCL = \bar{C} + 3\sqrt{\bar{C}} = 9 + 9 = 18$

$LCL = \bar{C} - \sqrt{\bar{C}} = 9 - 9 = 0$; $CL = \bar{C} = 9$

**18.18** $\Sigma x = 829.2$; $\Sigma R = 716$; $\bar{x} = \dfrac{\Sigma x}{n} = 71.6$;

$R = 87 - 42 = 45$;

$UCL_{\bar{x}} = \bar{x} + A_2\bar{R} = 106.024$;

$LCL_{\bar{x}} = \bar{x} - A_2\bar{R} = 37.27$;

$\bar{R} = \dfrac{\Sigma R}{n} = 59.66$; $UCL_R = 176.18$;

$LCL_R = 0$; $CL_R = 59.66$

**18.20** Total no. of defects = 87, sample size = 30

$\bar{C} = \dfrac{77}{30} = 2.57$;

$UCL = \bar{C} + 3\sqrt{\bar{C}} = 2.57 + 3\sqrt{2.57}$
$= 2.57 + 4.81 = 7.38$

$LCL = \bar{C} - 3\sqrt{\bar{C}} = 2.57 - 3\sqrt{2.57}$
$= 2.57 - 4.81 = -2.24$ or 0

**18.21** $\bar{C} = \dfrac{\Sigma C}{n} = \dfrac{35}{20} = 1.75$;

$UCL = \bar{C} + 3\sqrt{\bar{C}} = 1.75 + 3\sqrt{1.75} = 5.719$

$LCL = \bar{C} - 3\sqrt{\bar{C}} = 1.75 - 3\sqrt{1.75} = -2.219$ or 0

**18.22** $\bar{p} = \dfrac{\text{Total defectives}}{\text{Number of items inspected}} = \dfrac{169}{15 \times 100}$
$= 0.113$

$CL = \bar{p} = 0.113$;

$UCL = \bar{p} + 3\sqrt{\dfrac{\bar{p}(1-\bar{p})}{n}}$

$= 0.113 + 3\sqrt{\dfrac{0.113 \times 0.887}{100}} = 0.208$

$LCL = \bar{p} - 3\sqrt{\dfrac{\bar{p}(1-\bar{p})}{n}}$

$= 0.113 - 3\sqrt{\dfrac{0.113 \times 0.887}{100}} = 0.01$

## Case Studies

### Case 18.1: Vishal Chemical Company

In the molding operation of a plastic base for an electrical component, the most important characteristic was considered to be the length of the base, since this was required to fit snugly into the final assembly.

A sampling of the production process yielded the best results shown in Table 18.4 showing the length dimension in terms of a ten-thousandth of an inch plus the basic dimension of 1.6000 inches.

#### Questions for Discussion

1. Is the production process under control, as regards fluctuations in average level and variability?
2. What tolerances for base length is the present production process capable of holding?

**Table 18.5: Plastic Base Lengths**

Measurements in Ten-Thousandth of an Inch Plus the Basic Dimension of 1.6000 Inch.**

| *Sample* | *December* | | | | | |
|---|---|---|---|---|---|---|
| *Number* | *1* | *2* | *3* | *4* | *5* | *6* |
| 1 | 504 * | 503 | 481 | 482 | 489 | 469 |
| 2 | 458 * | 496 | 471 | 499 * | 479 | 489 |
| 3 | 479 | 481 * | 455 * | 462 * | 478 | 462 * |
| 4 | 478 | 504 * | 487 * | 475 | 468 | 495 * |
| 5 | 477 | 482 | 487 | 481 | 467 * | 484 |
| Total | 2396 | 2466 | 2381 | 2399 | 2381 | 2399 |
| Average | 479 | 493 | 476 | 480 | 476 | 480 |
| Range | 46 | 23 | 32 | 37 | 22 | 33 |

| *Sample Number* | *December* | | | | | |
|---|---|---|---|---|---|---|
| | *8* | *9* | *10* | *11* | *12* | *13* |
| 1 | 489 | 477 | 466 | 493 | 484 | 478 |
| 2 | 487 | 478 * | 480 | 464 * | 461 * | 476 * |
| 3 | 466 | 478 | 496 * | 479 | 481 | 477 |
| 4 | 507 | 478 | 459 * | 510 * | 480 | 494 * |
| 5 | 506 | 460 * | 481 | 475 | 514 | 476 |
| Total | 2455 | 2371 | 2382 | 2421 | 2420 | 2401 |
| Average | 491 | 474 | 476 | 484 | 484 | 480 |
| Range | 41 | 18 | 37 | 46 | 53 | 18 |

| *Sample Number* | *December* | | | | | |
|---|---|---|---|---|---|---|
| | *15* | *16* | *17* | *18* | *19* | *20* |
| 1 | 468 | 479 * | 490 | 478 | 469 | 451 |
| 2 | 449 * | 491 | 484 | 485 | 476 | 467 |
| 3 | 470 | 475 | 487 | 477 * | 457 * | 495 |
| 4 | 489 | 480 | 468 * | 480 | 466 | 484 |
| 5 | 493 * | 502 * | 497 * | 496 * | 477 * | 513 * |
| Total | 2369 | 2418 | 2426 | 2416 | 2345 | 2410 |
| Average | 474 | 484 | 485 | 483 | 469 | 482 |
| Range | 54 | 32 | 29 | 19 | 20 | 6 |

* High or low test result.

** For example, the entry value 504 = 1.6504 inch.

## Case 18.2: City Corporation

When using purchase requisitions in conjunction with an automated purchase-order processing system, it is essential that requisition forms be filled out fully and properly. Failure on the part of requisitioning personnel to comply with the form's requirements can lead to costly delays, since the computer section must then check back with the persons concerned, regarding missing or erroneous entries on the form. At other times, when an error is not caught in time, problems may arise after the material, parts, or supplies have been ordered, or when they are received, or later in actual production.

As part of a general administrative paperwork control system, one hundred requisitions were selected at random and checked each week. The results of this check are shown in Table 18.5. A 'defect', or faulty entry, would be concerned with an omission or an unreadable or erroneous entry on a requisition form.

Error could arise with regard to all sections of the form, particularly the part number and name, price quotation and date, buyer, material code, lead time, vendor name and number, shipping instructions, packaging specifications, quantities, terms, and other designations pertinent to adequate purchase requisitioning.

**Table 18.6: Analysis of Purchase Requistions**

| *Week Numbers* | *Defects Found in Samples of 100 Requisitions* | | | |
|---|---|---|---|---|
| | *Omissions* | *Errors* | *Unreadable* | *Total* |
| 1 | 1 | 0 | 1 | 2 |
| 2 | 3 | 0 | 2 | 5 |
| 3 | 1 | 1 | 0 | 2 |
| 4 | 8 | 2 | 3 | 13 |
| 5 | 2 | 0 | 0 | 2 |
| 6 | 0 | 1 | 1 | 2 |
| 7 | 6 | 2 | 2 | 10 |
| 8 | 2 | 1 | 0 | 3 |
| 9 | 0 | 0 | 1 | 1 |
| 10 | 4 | 1 | 4 | 9 |
| 11 | 4 | 1 | 3 | 8 |
| 12 | 1 | 0 | 0 | 1 |
| 13 | 0 | 0 | 0 | 0 |
| 14 | 0 | 1 | 2 | 3 |
| 15 | 2 | 1 | 0 | 3 |
| 16 | 1 | 2 | 0 | 3 |
| 17 | 0 | 2 | 0 | 2 |
| 18 | 2 | 1 | 1 | 4 |
| 19 | 1 | 0 | 2 | 3 |
| 20 | 2 | 0 | 2 | 4 |
| | 40 | 16 | 24 | 80 |

## Questions for Discussion

Evaluate the data by means of control charts for the various types of defects found, including the total weekly defects. Prepare recommendations for management concerning the control of deficiencies in the preparation of purchase requisitions.

Take time to deliberate, but when the time for action arrives, stop thinking and go on.

—Andrew Jackson

Chapter 19

# Statistical Decision Theory

## LEARNING OBJECTIVES

After studying this chapter, you should be able to

- identify decision alternatives, the states of nature and payoff associated with every possible combination of decision alternatives and states of nature.
- revise subjective probability values included in the formulation of the decision problem.
- identify the one best decision alternative for decision situation of uncertainty and risk.
- develop managerial judgement to utilize the subjective as well as objective interpretation of probabilities in decision making.

## 19.1 INTRODUCTION

Statistical methods discussed so far to draw inferences about population characteristics based on sample data, are also concerned with decision analysis. For example, the acceptance or rejection of a particular null hypothesis directly affects the managerial action or decision. However, in the data analysis described in this chapter, the decision alternatives available to a decision-maker are given due consideration. Making of a decision requires an enumeration of feasible and viable decision alternatives (courses of action or strategies), the projection of economic consequences associated with different alternatives, the use of subjective as well as objective interpretations of probabilities concerning random events, and a measure of effectiveness (of an objective) by which the most preferred decision alternative is identified. *Statistical decision theory* provides an analytical and systematic approach to the study of decision-making wherein data concerning the occurrence of different outcomes (consequences) may be evaluated to enable the decision-maker to identify the suitable decision alternative (or course of action).

Decision models that help decision-makers to arrive at the best possible decisions are classified according to the *degree of certainty*. The scale of certainty can range from complete certainty to complete uncertainty. The region which falls between these two extreme points corresponds to decision-making under risk (probabilistic problems).

Irrespective of the type of decision model, there are certain essential characteristics which are common to all as listed below.

**Decision alternatives:** There is a finite number of decision alternatives available with the decision-maker at each point in time when a decision is made. The number and type of such alternatives may depend on the previous decisions made and on what has happened subsequent to those decisions. These alternatives are also called *courses of action* (*actions*, *acts*, or *strategies*) and are under the control of and known to the decision-maker. These may be described numerically such as, stocking 100 units of a particular item, or non-numerically such as, conducting a market survey to know the likely demand of an item.

**States of nature:** Factors in a decision problem which affect the payoff of a decision and are beyond the control of the decision maker.

**State of nature:** A possible future condition (consequence or event) resulting from the choice of a decision alternative depends upon certain factors beyond the control of the decision-maker. These factors are called states of nature (future). For example, if the decision is to carry an umbrella or not, the consequence (get wet or do not) depends on what action nature takes.

The states of nature are mutually exclusive and collectively exhaustive with respect to any decision problem. The states of nature may be described numerically such as, demand of 100 units of an item or non-numerically such as, an employees strike.

**Payoff:** The consequences in a decision problem for any combination of states of nature and decision alternatives in terms of actual costs, profits, losses, gains, and so on.

**Payoff:** A numerical value resulting from each possible combination of alternatives and states of nature is called payoff. The payoff values are always conditional values because of unknown states of nature.

A tabular arrangement of these conditional outcome (payoff) values is known as *payoff matrix*, as shown in Table 19.1.

**Table 19.1: General Form of Payoff Matrix**

| | *Courses of Action (Alternatives)* | | | |
|---|---|---|---|---|
| *States of Nature* | $S_1$ | $S_2$ | ... | $S_n$ |
| $N_1$ | $p_{11}$ | $p_{12}$ | ... | $p_{1n}$ |
| $N_2$ | $p_{21}$ | $p_{22}$ | ... | $p_{2n}$ |
| ⋮ | ⋮ | ⋮ | | ⋮ |
| $N_m$ | $p_{m1}$ | $p_{m2}$ | ... | $p_{mn}$ |

## 19.2 STEPS IN DECISION THEORY APPROACH

The decision-making process involves the following steps:

1. Identifying and defining the problem.
2. Listing of all possible future events, called *states of nature*, which can occur in the context of the decision problem. Such events are not under the control of the decision-maker because these are erratic in nature.
3. Identification of all the *courses of action* (alternatives or decision choices) which are available to the decision-maker. The decision-maker has control over these courses of action.
4. Expressing the payoffs ($p_{ij}$) resulting from each pair of course of action and state of nature. These payoffs are normally expressed as a monetary value.
5. Apply an appropriate mathematical decision theory model to select the best course of action from the given list on the basis of some criterion (measure of effectiveness) that results in the optimal (desired) payoff.

**Example 19.1:** A firm manufactures three types of products. The fixed and variable costs are given below:

| | *Fixed Cost (Rs)* | *Variable Cost per Unit (Rs)* |
|---|---|---|
| Product A : | 25,000 | 12 |
| Product B : | 35,000 | 9 |
| Product C : | 53,000 | 7 |

The likely demand (units) of the products is given below:

| | | |
|---|---|---|
| Poor demand | : | 3000 |
| Moderate demand | : | 7000 |
| High demand | : | 11,000 |

If the sale price of each type of product is Rs 25, then prepare the payoff matrix.

**Solution:** Let $D_1$, $D_2$, and $D_3$ be the poor, moderate, and high demand, respectively. Then payoff is given by

$$\text{Payoff} = \text{Sales revenue} - \text{Cost}$$

The calculations for payoff (in thousand) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

$D_1 A = (3 \times 25) - 25 - (3 \times 12) = 14$

$D_2 A = (7 \times 25) - 25 - (7 \times 12) = 66$

$D_1 B = (3 \times 25) - 35 - (3 \times 9) = 13$

$D_2 B = (7 \times 25) - 35 - (7 \times 9) = 77$

$D_1 C = (3 \times 25) - 53 - (3 \times 7) = 1$

$D_2 C = (7 \times 25) - 53 - (7 \times 7) = 73$

$D_3 A = (11 \times 25) - 25 - (11 \times 12) = 118$

$D_3 B = (11 \times 25) - 35 - (11 \times 9) = 141$

$D_3 C = (11 \times 25) - 53 - (11 \times 7) = 145$

The payoff values are shown in Table 19.2.

**Table 19.2: (Rs in '000)**

| *Product Type* | *Alternative Demand* | | |
|---|---|---|---|
| | $D_1$ | $D_2$ | $D_3$ |
| A | 14 | 66 | 118 |
| B | 13 | 77 | 141 |
| C | 1 | 73 | 145 |

## 19.3 TYPES OF DECISION-MAKING ENVIRONMENTS

Decisions are made based upon the data available about the occurrence of events as well as the decision situation (or environment). There are four types of decision-making environments: *Certainty*, *uncertainty*, *risk*, and *conflict*.

**Decision-Making under Certainty** In this case the decision-maker has complete knowledge (perfect information) of the consequence of every decision choice (course of action or alternative) with certainty. Obviously, he will select an alternative that yields the largest return (payoff ) for the known future (state of nature). For example, the decision to purchase either N.S.C. (National Saving Certificate), Indira Vikas Patra, or deposit in N.S.S. (National Saving Scheme) is one in which it is reasonable to assume complete information about the future because there is no doubt that the Indian government will pay the interest when it is due and the principal at maturity.

In this decision model, only one possible state of nature (future) exists.

**Decision-Making under Risk** In this case the decision-maker has less than complete knowledge with certainty of the consequence of every decision choice (course of action). This means there is more than one state of nature (future) and for which he makes an assumption of the probability with which each state of nature will occur. For example, the probability of getting a head in the toss of a coin is 0.5.

**Decision-making under risk:** Decision problems in which the probability distribution on the states of nature is based on empirical data.

**Decision-Making under Uncertainty** In this case the decision-maker is unable to specify the probabilities with which the various states of nature (future) will occur. Thus, decisions under uncertainty are taken with even less information than decisions under risk. For example, the probability that Mr X will be the prime minister of the country 15 years from now is not known.

## 19.4 DECISION-MAKING UNDER UNCERTAINTY

**Decision-making under uncertainty:** Decision problems in which the probability distribution on the states of nature is based on subjective information.

In the absence of knowledge about the probability of any state of nature (future) occurring, the decision-maker must arrive at a decision only on the actual conditional payoff values, together with a policy (attitude). There are several different criteria of decision-making in this situation. The criteria that we will discuss in this section include:

(i) Maximax or Minimin
(ii) Maximin or Minimax
(iii) Equally Likely
(iv) Criterion of Realism
(v) Criterion of Regret

### 19.4.1 Criterion of Optimism (Maximax or Minimin)

In this criterion the decision-maker ensures that he should not miss the opportunity to achieve the largest possible profit (maximax) or lowest possible cost (minimin). Thus, he selects the alternative (decision choice or course of action) that represents the maximum of the maxima (or minimum of the minima) payoffs (consequences or outcomes). The working method is summarized as follows:

(a) Locate the maximum (or minimum) payoff values corresponding to each alternative (or course of action), then
(b) Select an alternative with the best anticipated payoff value (maximum for profit and minimum for cost).

Since in this criterion the decision-maker selects an alternative with the largest (or lowest) possible payoff value, it is also called an *optimistic decision criterion.*

### 19.4.2 Criterion of Pessimism (Maximin or Minimax)

In this criterion the decision-maker ensures that he would earn no less (or pay no more) than some specified amount. Thus, he selects the alternative that represents the maximum of the minima (or mimimum of the maxima in case of loss) payoff in case of profits. The working method is summarized as follows:

(a) Locate the minimum (or maximum in case of profit) payoff value in case of loss (or cost) data corresponding to each alternative, then
(b) Select an alternative with the best anticipated payoff value (maximum for profit and mimimum for loss or cost).

Since in this criterion the decision-maker is conservative about the future and always anticipates the worst possible outcome (minimum for profit and maximum for cost or loss), it is called a *pessimistic decision criterion*. This criterion is also known as *Wald's criterion*.

### 19.4.3 Equally Likely Decision (Laplace) Criterion

Since the probabilities of the states of nature are not known, it is assumed that all states of nature will occur with equal probability, that is, each state of nature is assigned an equal probability. As states of nature are mutually exclusive and collectively exhaustive, so the probability of each of these must be 1/(number of states of nature). The working method is summarized as follows:

(a) Assign equal probability value to each state of nature by using the formula:

*1 ÷ (Number of states of nature)*

(b) Compute the expected (or average) payoff for each alternative (course of action) by adding all the payoffs and dividing by the number of possible states of nature or by applying the formula:

*(Probability of state of nature j)* × *(Payoff value for the combination of alternative i and state of nature j)*

(c) Select the best expected payoff value (maximum for profit and minimum for cost).

This criterion is also known as the criterion of insufficient reason because, except in a few cases, some information of the likelihood of occurrence of the states of nature is available.

### 19.4.4 Criterion of Realism (Hurwicz Criterion)

This criterion suggests that a rational decision-maker should be neither completely optimistic nor pessimistic and, therefore, must display a mixture of both. Hurwicz, who suggested this criterion, introduced the idea of a coefficient of optimism (denoted by $\alpha$) to measure the decision-maker's degree of optimism. This coefficient lies between 0 and 1, where 0 represents a complete pessimistic attitude about the future and 1 a complete optimistic attitude about the future. Thus, if $\alpha$ is the coefficient of optimism, then $(1 - \alpha)$ will represent the coefficient of pessimism.

The Hurwicz approach suggests that the decision maker must select an alternative that maximizes:

$$H(\textit{Criterion of realism}) = \alpha(\textit{Maximum in column}) + (1 - \alpha)(\textit{Minimum in column})$$

The working method is summarized as follows:

(a) Decide the coefficient of optimism $\alpha$ (alpha) and then the coefficient of pessimism $(1 - \alpha)$.
(b) For each alternative select the largest and smallest payoff values and multiply these with $\alpha$ and $(1 - \alpha)$ values, respectively. Then calculate the weighted average, H by using the above formula.
(c) Select an alternative with the best anticipated weighted average payoff value.

### 19.4.5 Criterion of Regret

This criterion is also known as the opportunity loss decision criterion or minimax regret decision criterion because the decision-maker feels regret after adopting a wrong course of action (or alternative) resulting in an opportunity loss of payoff. Thus, he always intends to minimize this regret. The working method is summarized as follows:

(a) From the given payoff matrix, develop an opportunity-loss (or regret) matrix as follows:
    (i) Find the best payoff corresponding to each state of nature, and
    (ii) Subtract all other entries (payoff values) in that row from this value.
(b) For each course of action (strategy or alternative) identify the worst or maximum regret value. Record this number in a new row.
(c) Select the course of action (alternative) with the smallest anticipated opportunity-loss value.

**Example 19.2:** A food products company is contemplating the introduction of a revolutionary new product with new packaging, or replace the existing product at a much higher price ($S_1$), or a moderate change in the composition of the existing product with a new packaging at a small increase in price ($S_2$), or a small change in the composition of the existing product except the word 'New' with a negligible increase in price ($S_3$). The three possible states of nature or events are: (i) high increase in sales ($N_1$), (ii) no change in sales ($N_2$), and (iii) decrease in sales ($N_3$). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of the three events (expected sales). This is represented in the following table:

| | *States of Nature* | | |
|---|---|---|---|
| *Strategies* | $N_1$ | $N_2$ | $N_3$ |
| $S_1$ | 7,00,000 | 3,00,000 | 1,50,000 |
| $S_2$ | 5,00,000 | 4,50,000 | 0 |
| $S_3$ | 3,00,000 | 3,00,000 | 3,00,000 |

Which strategy should the concerned executive choose on the basis of

(a) Maximin criterion
(b) Maximax criterion
(c) Minimax regret criterion
(d) Laplace criterion?

**Solution:** The payoff matrix is rewritten as follows:

(a) *Maximin Criterion*

| *States of Nature* | *Strategies* $S_1$ | $S_2$ | $S_3$ | |
|---|---|---|---|---|
| $N_1$ | 7,00,000 | 5,00,000 | 3,00,000 | |
| $N_2$ | 3,00,000 | 4,50,000 | 3,00,000 | |
| $N_3$ | 1,50,000 | 0 | 3,00,000 | |
| Column minimum | 1,50,000 | 0 | **3,00,000** | ← *Maximin* |

The maximum of column minima is 3,00,000. Hence, the company should adopt strategy $S_3$.

(b) *Maximax Criterion*

| *States of Nature* | *Strategies* $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $N_1$ | 7,00,000 | 5,00,000 | 3,00,000 |
| $N_2$ | 3,00,000 | 4,50,000 | 3,00,000 |
| $N_3$ | 1,50,000 | 0 | 3,00,000 |
| Column maximum | **7,00,000** | 5,00,000 | 3,00,000 |
| | ↑ *Maximax* | | |

The maximum of column maxima is 7,00,000. Hence, the company should adopt strategy $S_1$.

(c) *Minimax Regret Criterion* Opportunity loss table is shown below:

| *States of Nature* | *Strategies* $S_1$ | $S_2$ | $S_3$ |
|---|---|---|---|
| $N_1$ | 7,00,000 – 7,00,000 = 0 | 7,00,000 – 5,00,000 = 2,00,000 | 7,00,000 – 3,00,000 = 4,00,000 |
| $N_2$ | 4,50,000 – 3,00,000 = 1,50,000 | 4,50,000 – 4,50,000 = 0 | 4,50,000 – 3,00,000 = 1,50,000 |
| $N_3$ | 3,00,000 – 1,50,000 = 1,50,000 | 3,00,000 – 0 = 3,00,000 | 3,00,000 – 3,00,000 = 0 |
| Column maximum | **1,50,000** | 3,00,000 | 4,00,000 |
| | ↑ *Minimax regret* | | |

Hence the company should adopt the minimum opportunity loss strategy, $S_1$.

(d) *Laplace Criterion* Since we do not know the probabilities of the states of nature, assume that they are equal. For this example, we would assume that each state of nature has a one third probability of occurrence. Thus,

| *Strategy* | *Expected Return (Rs)* |
|---|---|
| $S_1$ | (7,00,000 + 3,00,000 + 1,50,000)/3 = 3,83,333.33 |
| $S_2$ | (5,00,000 + 4,50,000 + 0)/3 = 3,16,666.66 |
| $S_3$ | (3,00,000 + 3,00,000 + 3,00,000)/3 = 3,00,000 |

Since the largest expected return is from strategy $S_1$, the executive must select strategy $S_1$.

**Example 19.3:** A manufacturer makes a product, of which the principal ingredient is a chemical, X. At the moment, the manufacturer spends Rs 1000 per year on supply of X, but there is a possibility that the price may soon increase to four times its present figure because of a worldwide shortage of the chemical. There is another chemical Y, which the manufacturer could use in conjunction with a third chemical, Z in order to give the same effect as chemical X. Chemicals Y and Z would together cost the manufacturer Rs 3000 per year, but their prices are unlikely to rise. What action should the manufacturer take? Apply the maximin and minimax criteria for decision-making and give two sets of solutions. If the coefficent of optimism is 0.4, find the course of action that minimizes the cost. *[ICWA, Dec., 1998]*

**Solution:** The data of the problem is summarized in the following table (negative figures in the table represent profit).

| *States of Nature* | *Courses of Action* | |
|---|---|---|
| | $S_1$ *(use Y and Z)* | $S_2$ *(use X)* |
| $N_1$ (Price of X increases) | – 3000 | – 4000 |
| $N_2$ (Price of X does not increase) | – 3000 | – 1000 |

(a) *Maximin Criterion*

| *States of Nature* | *Courses of Action* | |
|---|---|---|
| | $S_1$ | $S_2$ |
| $N_1$ | – 3000 | – 4000 |
| $N_2$ | – 3000 | – 1000 |
| Column minimum | **– 3000** | – 4000 |
| | ↑ *Maximin* | |

Maximum of column minima = – 3,000. Hence, the manufacturer should adopt action $S_1$.

(b) *Minimax (or opportunity loss) Criterion*

| *States of Nature* | *Courses of Action* | |
|---|---|---|
| | $S_1$ | $S_2$ |
| $N_1$ | – 3000 – (– 3000) = 0 | – 3000 – (– 4000) = 1000 |
| $N_2$ | – 1000 – (– 3000) = 2000 | – 1000 – (– 1000) = 0 |
| Maximum opportunity | 2000 | **1000** ← *Minimax* |

Hence, the manufacturer should adopt the minimum opportunity loss course of action $S_2$.

(c) *Hurwicz Criterion:* Given the coefficient of optimism equal to 0.4, the coefficient of pessimism will be 1 – 0.4 = 0.6. Then according to Hurwicz, a select course of action that optimizes (maximum for profit and minimum for cost) the payoff value

$$H = \alpha \text{ (Best payoff)} + (1 - \alpha) \text{ (Worst payoff)}$$
$$= \alpha \text{ (Maximum in column)} + (1 - \alpha) \text{ (Minimum in column)}$$

| *Course of Action* | *Best Payoff* | *Worst Payoff* | *H* |
|---|---|---|---|
| $S_1$ | – 3000 | – 3000 | – 3000 |
| $S_2$ | – 1000 | – 4000 | – 2800 |

Since course of action $S_2$ has the least cost (maximum profit) = 0.4(1000) + 0.6(4000) = Rs 2800, the manufacturer should adopt it.

## Self-Practice Problems 19A

**19.1** The following matrix gives the payoff (in Rs) of different strategies (alternatives) $S_1$, $S_2$, and $S_3$ against conditions (events) $N_1$, $N_2$, $N_3$, and $N_4$.

| *Strategy* | *States of Nature* | | | |
|---|---|---|---|---|
| | $N_1$ | $N_2$ | $N_3$ | $N_4$ |
| $S_1$ | 4000 | – 100 | 6000 | 18,000 |
| $S_2$ | 20,000 | 5000 | 400 | 0 |
| $S_3$ | 20,000 | 15,000 | – 2000 | 1000 |

Indicate the decision taken under the following approaches: (a) Pessimistic, (b) Optimistic, (c) Equal probability, (d) Regret, (e) Hurwicz criterion. The degree of optimism being 0.7.

**19.2** In a toy manufacturing company, suppose the product acceptance probabilities are not known, but the following data is known:

| Product Acceptance | Anticipated First Year Profit (Rs in '000s) Product Line: Full | Partial | Minimal |
|---|---|---|---|
| Good | 8 | 70 | 50 |
| Fair | 50 | 45 | 40 |
| Poor | - 25 | - 10 | 0 |

Determine the optimal decision under each of the following decision criteria and show how you arrived at it: (a) Maximax, (b) Maximin, (c) Equal likelihood, and (d) Minimax regret?

**19.3** The following is a payoff (in rupees) table for three strategies and two states of nature:

| Strategy | State of Nature $N_1$ | $N_2$ |
|---|---|---|
| $S_1$ | 40 | 60 |
| $S_2$ | 10 | - 20 |
| $S_3$ | - 40 | 150 |

Select a strategy using each of the following decision criteria: (a) Maximax, (b) Minimax regret, (c) Maximin, (d) Minimum risk, assuming equiprobable states.

**19.4** Mr Sethi has Rs 10,000 to invest in one of three options, A, B, or C. The return on his investment depends on whether the economy experiences inflation, recession, or no change at all. His possible returns under each economic condition are given below:

| Strategy | State of Nature: Inflation | Recession | No Change |
|---|---|---|---|
| A | 2000 | 1200 | 1500 |
| B | 3000 | 800 | 1000 |
| C | 2500 | 1000 | 1800 |

What would be his decision using the pessimistic criterion, optimistic criterion, equally likely criterion, and regret criterion?

**19.5** A manufacturer's representative has been offered a new product line. If he accepts the new line he can handle it in one of the two ways. The best way according to the manufacturer would be to have a separate sales force to handle the new line exclusively. This would involve an initial investment of Rs 1,00,000 in the office, equipment, and the hiring and training of the salesmen. On the other hand, if the new line could be handled by the existing sales force using the existing facilities, the initial investment would be only Rs 30,000, principally for training of his present salesmen.

The new product sells for Rs 250. The representative normally receives 20 per cent of the sale price on each unit sold of which 10 per cent is paid as commission to handle the new product. The manufacturer offers to pay 60 per cent of the sale price of each unit sold to the representative if the representative sets up a separate sales organization. Otherwise the normal 20 per cent will be paid. In either case the salesman gets a 10 per cent commission. Based on the size of the territory and experience with other products, the representative estimates the probabilities of annual sales of the new product:

| Sales (in units) | Probability |
|---|---|
| 1000 | 0.10 |
| 2000 | 0.15 |
| 3000 | 0.40 |
| 4000 | 0.30 |
| 5000 | 0.05 |

(a) Set up a regret table.
(b) Find the expected regret of each course of action.
(c) Which course of action would have been best under the maximin criterion?

## Hints and Answers

**19.1** (a) $S_2$, (b) $S_2$ or $S_3$, (c) $S_3$, (d) $S_1$, (e) $S_2$.
**19.2** (a) Full, (b) Minimal, (c) Full or partial, (d) Partial.
**19.3** (a) $S_3$; Rs 150 (b) $S_3$; Rs 80 (c) $S_1$; Rs 40 (d) $S_3$; Rs 55.
**19.4** Choose A: Rs 120, Choose B: Rs 300, Choose C: Rs 176.6, Choose C: Rs 50.
**19.5** Let $S_1$ = install new sales facilities
$S_2$ = continue with existing sales facilities.
Therefore, payoff function corresponding to $S_1$ and $S_2$ would be:

$$S_1 = -1{,}00{,}000 + 250 \times \frac{(30-10)}{100} \times \alpha$$
$$= -1{,}00{,}000 + 50\alpha$$
$$S_2 = -30{,}000 + 250 \times \frac{(20-10)}{100} \times \alpha$$
$$= -30{,}000 + 25\alpha$$

Equating the two, we get $-1{,}00{,}000 + 50\alpha = -30{,}000 + 25\alpha$ or $\alpha = 2800$.

## 19.5 DECISION-MAKING UNDER RISK

Decision-making under risk is a probabilistic decision situation in which more than one state of nature exists and the decision-maker has sufficient information to assign probability values to the likely occurrence of each of these states. Knowing the probability distribution of the states of nature, the best decision is to select that course of action which has the largest expected payoff value.

The most widely used criterion for evaluating various courses of action (alternatives) under risk is the *Expected Monetary Value* (EMV) or *Expected Utility*.

### 19.5.1 Expected Monetary Value (EMV)

The expected monetary value (EMV) for a given course of action is the weighted average payoff, which is the sum of the payoffs for each course of action multiplied by the probabilities associated with each state of nature. Mathematically EMV is stated as follows:

$$\text{EMV } (\textit{Course of action, } S_j) = \sum_{i=1}^{m} p_{ij}\, p_i$$

where $m$ = number of possible states of nature

$p_i$ = probability of occurrence of state of nature $i$

$p_{ij}$ = payoff associated with state of nature $N_i$ and course of action, $S_j$

**Steps for Calculating EMV** The various steps involved in the calculation of EMV are as follows:

(a) Construct a payoff matrix listing all possible courses of action and states of nature. Enter the conditional payoff values associated with each possible combination of course of action and state of nature along with the probabilities of the occurrence of each state of nature.

(b) Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and add these weighted values for each course of action.

(c) Select the course of action that yields the optimal EMV.

**Example 19.4:** Mr X often flies from town A to town B. He can use the airport bus which costs Rs 25 but if he takes it, there is a 0.08 chance that he will miss the flight. The stay in a hotel costs Rs 270 with a 0.96 chance of being on time for the flight. For Rs 350 he can use a taxi which has a 99 per cent chance of being on time for the flight. If Mr X catches the plane on time, he will conclude a business transaction which will produce a profit of Rs 10,000, otherwise he will lose it. Which mode of transport should Mr X use? Answer on the basis of the EMV criterion. [*CA, May 1990*]

**Solution:** Computation of EMV of the various courses of action is shown in Table 19.3.

**Table 19.3**

| *States of Nature* | *Courses of Action* | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | *Bus* | | | *Stay in Hotel* | | | *Taxi* | | |
| | *Cost* | *Prob.* | *Expected Value* | *Cost* | *Prob.* | *Expected Value* | *Cost* | *Prob.* | *Expected Value* |
| Catches the flight | 10,000 − 25 = 9975 | 0.92 | 9177 | 10,000 − 270 = 9730 | 0.96 | 9,340.80 | 10,000 − 350 = 9650 | 0.99 | 9553.50 |
| Miss the flight | −25 | 0.08 | −2.0 | −270 | 0.04 | −10.80 | −350 | 0.01 | −3.50 |
| Expected monetary value (EMV) | | | 9175 | | | 9330 | | | 9330 |

Comparing the EMV associated with each course of action indicates that course of action 'Taxi' is the logical alternative because it has the highest EMV.

**Example 19.5:** The manager of a flower shop promises his customers delivery within four hours on all orders. All flowers are purchased the previous day and delivered to Parker by 8.00 a.m. the next morning. The daily demand for roses is as follows:

| | | | | | |
|---|---|---|---|---|---|
| Dozens of roses | : | 70 | 80 | 90 | 100 |
| Probability | : | 0.1 | 0.2 | 0.4 | 0.3 |

The manager purchases roses for Rs 10 per dozen and sells them for Rs 30. All unsold roses are donated to a local hospital. How many dozens of roses should Parker order each evening to maximize its profits? What is the optimum expected profit?

*[Delhi Univ., MBA, Dec. 1986, 2002]*

**Solution:** Since the number of roses (in dozens) purchased is under control of decision-maker, purchase per day is considered as 'course of action' (decision choice) and the daily demand of the flowers is uncertain and only known with probability, therefore, it is considered as a 'state of nature' (event). From the data, it is clear that the flower shop must not purchase less than 7 or more than 10 dozen roses per day. Also, each dozen roses sold within a day yields a profit of Rs (30 – 10) = Rs 20, otherwise it is a loss of Rs 10. Thus

Marginal profit (MP) = Selling price – Cost = 30 – 10 = Rs 20

Marginal loss (ML) = Loss on unsold roses = Rs 10

Using the information given in the problem, the various conditional profit (payoff) values for each combination of decision choice-event are given by

Conditional profit = MP × Roses sold – ML × Roses not sold

$$= \begin{cases} 20D, & \text{if } D \geq S \\ 20D - 10(S - D) = 30D - 10S, & \text{if } D < S \end{cases}$$

where D denotes the number of roses sold within a day and S the number of roses stocked.

The resulting conditional profit values and corresponding expected payoffs are computed in Table 19.4.

**Table 19.4: Conditional Profit Value (Payoffs)**

| *States of Nature (Demand per Day)* | *Probability* | *Conditional Profit (Rs) due to Courses of Action (Purchase per Day)* | | | | *Expected Payoff (Rs) due to Courses of Action (Purchase per Day)* | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | *70* | *80* | *90* | *100* | *70* | *80* | *90* | *100* |
| | *(1)* | *(2)* | *(3)* | *(4)* | *(5)* | *(1) × (2)* | *(1) × (3)* | *(1) × (4)* | *(1) × (5)* |
| 70 | 0.1 | 140 | 130 | 120 | 110 | 14 | 13 | 12 | 11 |
| 80 | 0.2 | 140 | 160 | 150 | 140 | 28 | 32 | 30 | 28 |
| 90 | 0.4 | 140 | 160 | 180 | 170 | 56 | 64 | 72 | 68 |
| 100 | 0.3 | 140 | 160 | 180 | 200 | 42 | 48 | 54 | 60 |
| Expected monetary value (EMV) | | | | | | 140 | 157 | 168 | 167 |

Since the highest EMV of Rs 168 is corresponding to course of action 90, the flower shop should purchase nine dozen roses everyday.

**Example 19.6:** A retailer purchases cherries every morning at Rs 50 a case and sells them for Rs 80 a case. Any case remaining unsold at the end of the day can be disposed of the next day at a salvage value of Rs 20 per case (thereafter they have no value). Past sales have ranged from 15 to 18 cases per day. The following is the record of sales for the past 120 days:

| | | | | | |
|---|---|---|---|---|---|
| Cases sold | : | 15 | 16 | 17 | 18 |
| Number of days | : | 12 | 24 | 48 | 36 |

Find how many cases the retailer should purchase per day to maximize his profit.

*[Delhi Univ., MCom, 1985; Ajmer MBA, 1988]*

**Solution:** Let $N_i$ ($i$ = 1, 2, 3, 4) be the possible states of nature (daily likely demand) and $S_j$ ($j$ = 1, 2, 3, 4) be all possible courses of action (number of cases of cherries to be purchased).

Marginal profit (MP) = Selling price – Cost = Rs (80 – 50) = Rs 30

Marginal Loss (ML) = Loss on unsold cases = Rs (50 – 20) = Rs 30

The conditional profit (payoff) values for each act-event combination are given by:

$$\begin{aligned}\text{Conditional profit} &= \text{MP} \times \text{Cases sold} - \text{ML} \times \text{Cases unsold}\\ &= (80 - 50)\,(\text{Cases sold}) - (50 - 20)\,(\text{Cases unsold})\\ &= 30S, \quad \text{if } S \geq N\\ &= (80 - 50)\,S - 30(N - S) = 60S - 30N \quad \text{if } S < N\end{aligned}$$

The resulting conditional profit values and corresponding expected payoffs are computed in Table 19.5.

**Table 19.5: Conditional Profit Values (Payoffs)**

| *States of Nature (Demand per Week)* | *Probability* | *Conditional Profit (Rs) due to Courses of Action (Purchase per Day)* | | | | *Expected Payoff (Rs) due to Courses of Action (Purchase per Day)* | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | *15* | *16* | *17* | *18* | *15* | *16* | *17* | *18* |
| | *(1)* | (2) | (3) | (4) | (5) | (1)×(2) | (1)×(3) | (1)×(4) | (1)×(5) |
| 15 | 0.1 | 450 | 420 | 390 | 360 | 45 | 42 | 39 | 36 |
| 16 | 0.2 | 450 | 480 | 450 | 420 | 90 | 96 | 90 | 84 |
| 17 | 0.4 | 450 | 480 | 510 | 480 | 180 | 192 | 204 | 192 |
| 18 | 0.3 | 450 | 480 | 510 | 540 | 135 | 144 | 153 | 162 |
| Expected monetary value (EMV) | | | | | | 450 | 474 | 486 | 474 |

Since the highest EMV of Rs 486 is corresponding to course of action 17, the retailer must purchase 17 cases of cherries every morning.

**Example 19.7:** The probability of the demand for lorries for hiring on any day in a given district is as follows:

| | | | | | |
|---|---|---|---|---|---|
| No. of lorries demanded : | 0 | 1 | 2 | 3 | 4 |
| Probability : | 0.1 | 0.2 | 0.3 | 0.2 | 0.2 |

Lorries have a fixed cost of Rs 90 each day to keep the daily hire charges (net of variable costs of running) Rs 200. If the lorry-hire company owns 4 lorries, what is its daily expectation? If the company is about to go into business and currently has no lorries, how many lorries should it buy? [*CA, May 1985*]

**Solution:** It is given that Rs 90 is the fixed cost and Rs 200 the variable cost. Now the payoff values with 4 lorries at the disposal of the decision-maker are calculated as under:

| | | | | | |
|---|---|---|---|---|---|
| No. of lorries demanded : | 0 | 1 | 2 | 3 | 4 |
| Payoff (with 4 lorries) : | 0 – 90 × 4 = – 360 | 200 – 90 × 4 = – 160 | 400 – 90 × 4 = 40 | 600 – 90 × 4 = 240 | 800 – 90 × 4 = 440 |

Thus daily expectation is obtained by multiplying the payoff values with the given corresponding probabilities of demand:

$$\begin{aligned}\text{Daily Expectation} &= (-360)(0.1) + (-160)(0.2) + (40)(0.3) + (240)(0.2) +\\ &\quad (440)(0.2) = \text{Rs } 80\end{aligned}$$

The conditional payoffs and expected payoffs for each course of action are shown in Tables 19.6 and 19.7.

**Table 19.6: Conditional Payoff Values**

| *Number of Lorries* | *Probability* | *Conditional Payoff (Rs) due to Decision to Purchase Lorries (Course of Action)* | | | | |
|---|---|---|---|---|---|---|
| | | *0* | *1* | *2* | *3* | *4* |
| 0 | 0.1 | 0 | − 90 | − 180 | − 270 | − 360 |
| 1 | 0.2 | 0 | 110 | 20 | − 70 | − 160 |
| 2 | 0.3 | 0 | 110 | 220 | 130 | 40 |
| 3 | 0.2 | 0 | 110 | 220 | 330 | 240 |
| 4 | 0.2 | 0 | 110 | 220 | 330 | 440 |

**Table 19.7: Expected Payoffs and EMV**

| *Number of Lorries* | *Probability* | *Conditional Payoff (Rs) due to Decision to Purchase Lorries (Course of Action)* | | | | |
|---|---|---|---|---|---|---|
| | | *0* | *1* | *2* | *3* | *4* |
| 0 | 0.1 | 0 | − 9 | − 18 | − 27 | − 36 |
| 1 | 0.2 | 0 | 22 | 4 | − 14 | − 32 |
| 2 | 0.3 | 0 | 33 | 66 | 39 | 12 |
| 3 | 0.2 | 0 | 22 | 44 | 66 | 48 |
| 4 | 0.2 | 0 | 22 | 44 | 66 | 88 |
| EMV | | 0 | 90 | 140 | 130 | 80 |

Since EMV of Rs 140 for the course of action 2 is highest, the company should buy 2 lorries.

### 19.5.2 Expected Opportunity Loss (EOL)

**Opportunity loss:** The absolute value of the difference between the payoff actually realized for a decision alternative and the payoff which could have been obtained had the optimal decision alternative been selected.

An alternative approach to maximizing expected monetary value (EMV) is to minimize the **expected opportunity loss** (EOL), also called *expected value of regret.* The EOL is defined as the difference between the highest profit (or payoff) for a state of nature and the actual profit obtained for the particular course of action taken. In other words, EOL is the amount of payoff that is lost by not selecting the course of action that has the greatest payoff for the state of nature that actually occurs. The course of action due to which EOL is minimum is recommended.

Since EOL is an alternative decision criterion for decision-making under risk, therefore, the results will always be the same as those obtained by EMV criterion discussed earlier. Thus, only one of the two methods should be applied to reach a decision. Mathematically, it is stated as follows:

$$\text{EOL (State of nature, } N_i) = \sum_{i=1}^{m} l_{ij}\, p_i$$

where $l_{ij}$ = opportunity loss due to state of nature, $N_i$ and course of action, $S_j$

$p_i$ = probability of occurrence of state of nature, $N_i$

**Steps for Calculating EOL** The various steps involved in the calculation of EOL are as follows:

1. Prepare a conditional profit table for each course of action and state of nature combination along with the associated probabilities.
2. For each state of nature calculate the conditional opportunity loss (COL) values by subtracting each payoff from the maximum payoff for that outcome.

3. Calculate EOL for each course of action by multiplying the probability of each state of nature with the COL value and then adding the values.
4. Select a course of action for which the EOL value is minimum.

**Example 19.8:** A company manufactures goods for a market in which the technology of the product is changing rapidly. The research and development department has produced a new product which appears to have the potential for commercial exploitation. A further Rs 60,000 is required for development testing.

The company has 100 customers and each customer might purchase at the most one unit of the product. Market research suggests that a selling price of Rs 6000 for each unit with total variable costs of manufacturing and selling estimate are Rs 2000 for each unit.

From previous experience, it has been possible to derive a probability distribution relating to the proportion of customers who will buy the product. It is as follows:

| | | | | | | |
|---|---|---|---|---|---|---|
| Proportion of customers | : | 0.04 | 0.08 | 0.12 | 0.16 | 0.20 |
| Probability | : | 0.10 | 0.10 | 0.20 | 0.40 | 0.20 |

Determine the expected opportunity losses, given no other information than that stated above, and state whether or not the company should develop the product.

**Solution:** If $p$ is the proportion of customers who purchase the new product, the conditional profit is: $(6000 - 2000) \times 100p - 60,000 =$ Rs $(4,00,000p - 60,000)$

Let $N_i$ ($i = 1, 2, \ldots, 5$) be the possible states of nature, that is, proportion of the customers who will buy the new product and $S_1$ (develop the product) and $S_2$ (do not develop the product) be the two courses of action.

The conditional profit values (payoffs) for each pair of $N_i$'s and $S_j$'s are shown in Table 19.8.

**Table 19.8: Conditional Profit Values (Payoffs)**

| *State of Nature (Proportion of Customers, p)* | *Conditional Profit = Rs (4,00,000 p – 60,000)* | |
|---|---|---|
| | *Course of Action* | |
| | *$S_1$ (Develop)* | *$S_2$ (Do not Develop)* |
| 0.04 | – 44,000 | 0 |
| 0.08 | – 28,000 | 0 |
| 0.12 | – 12,000 | 0 |
| 0.16 | 4000 | 0 |
| 0.20 | 20,000 | 0 |

Opportunity loss values are shown in Table 19.9.

**Table 19.9: Opportunity Loos Values**

| *State of Nature* | *Probability* | *Conditional Profit (Rs)* | | *Opportunity Loss (Rs)* | |
|---|---|---|---|---|---|
| | | *$S_1$* | *$S_2$* | *$S_1$* | *$S_2$* |
| 0.04 | 0.1 | – 44,000 | 0 | 44,000 | 0 |
| 0.08 | 0.1 | – 28,000 | 0 | 28,000 | 0 |
| 0.12 | 0.2 | – 12,000 | 0 | 12,000 | 0 |
| 0.16 | 0.4 | 4000 | 0 | 0 | 4000 |
| 0.20 | 0.2 | 20,000 | 0 | 0 | 20,000 |

Using the given estimates of probabilities associated with each state of nature, the expected opportunity loss (EOL) for each course of action is given below:

EOL $(S_1) = 0.1\ (44,000) + 0.1\ (28,000) + 0.2\ (12,000) + 0.4\ (0) + 0.2\ (0) =$ Rs 9600

EOL $(S_2) = 0.1\ (0) + 0.1\ (0) + 0.2\ (0) + 0.4\ (4000) + 0.2\ (20,000) =$ Rs 5600

Since the company seeks to minimize the expected opportunity loss, the company should select course of action $S_2$ (do not develop the product) with minimum EOL.

### 19.5.3 Expected Value of Perfect Information (EVPI)

In decision-making under risk each state of nature is associated with the probability of its occurrence. However, if the decision-maker can acquire *perfect* (complete and accurate) *information* about the occurrence of various states of nature, then he will be able to select a course of action that yields the desired payoff for whatever state of nature that actually occurs.

We have seen that the EMV or EOL criterion helps the decision-maker select a particular course of action that optimizes the expected payoff without any additional information. The *expected value of perfect information* (EVPI) represents the maximum amount of money the decision-maker has to pay to get this additional information about the occurrence of various states of nature before a decision has to be made. Mathematically it is stated as

EVPI = Expected profit (or value) with perfect information under certainty – Expected profit without perfect information

$$= \sum_{i=1}^{m} p_{ij}(N_i)p_i - EMV^*$$

where $p_{ij}$ = best payoff for the state of nature, $N_i$
$p_i$ = probability of the state of nature, $N_i$
$EMV^*$ = maximum expected monetary value

**Example 19.9:** A company needs to increase its production beyond its existing capacity. It has narrowed the alternatives to two approaches to do so: (a) expansion at a cost of Rs 8 million, or (b) modernization at a cost of Rs 5 million. Both approaches would require the same amount of time for implementation. Management believes that over the required payback period, demand will either be high or moderate. Since high demand is considered to be somewhat less likely than moderate demand, the probability of high demand has been set at 0.35. If the demand is high, expansion would gross an estimated additional Rs 12 million but modernization only an additional Rs 6 million, due to a lower maximum production capability. On the other hand, if the demand is moderate, the comparable figures would be Rs 7 million for expansion and Rs 5 million for modernization.

(a) Calculate the conditional profit in relation to various action-and-outcome combinations and states of nature.
(b) If the company wishes to maximize its expected monetary value (EMV), should it modernize or expand?
(c) Calculate the EVPI.
(d) Construct the conditional opportunity loss table and also calculate EOL.

[*Delhi Univ., MBA, 1998*]

**Solution:** (a) Defining the states of nature: High demand or moderate demand (over which the company has no control) and courses of action (company's possible decisions): Expand or Modernize.

Since the probability that the demand is high is estimated at 0.35, the probability of moderate demand must be (1 – 0.35) = 0.65. The calculations for conditional profit values are shown in Table 19.10.

**Table 19.10: Conditional Profit Table**

| *State of Nature (Demand)* | *Conditional Profit (Rs in million) due to Course of Action* | |
|---|---|---|
| | *Expand ($S_1$)* | *Modernize ($S_2$)* |
| High demand ($N_1$) | 12 – 8 = 4 | 6 – 5 = 1 |
| Moderate demand ($N_2$) | 7 – 8 = – 1 | 5 – 5 = 0 |

(b) The payoff table (Table 19.10) can be rewritten as follows along with the given probabilities of the states of nature.

**Table 19.11: Payoff Table**

| *State of Nature (Demand)* | *Probability* | *Conditional Profit (Rs in million) due to Course of Action* | |
|---|---|---|---|
| | | *Expand* | *Modernize* |
| High demand | 0.35 | 4 | 1 |
| Moderate demand | 0.65 | – 1 | 0 |

The calculation of EMV for each course of action $S_1$ and $S_2$ is given below:

$$EMV(S_1) = 0.35(4) + 0.65(-1) = \text{Rs } 0.75 \text{ million}$$

$$EMV(S_2) = 0.35(1) + 0.65(0) = \text{Rs } 0.35 \text{ million}$$

Thus to maximize EMV, the company must choose the course of action $S_1$ (expand). The EMV of the optimal course of action is generally denoted by EMV*. Therefore,

$$EMV^* = EMV(S_1) = \text{Rs } 0.75 \text{ million}$$

(c) To calculate EVPI, we shall first calculate EPPI. For calculating EPPI, we choose the optimal course of action for each state of nature, multiply its conditional profit by the given probability to get weighted profit, and then sum these weights as shown in Table 19.12.

**Table 19.12: Payoff Table**

| *State of Nature (Demand)* | *Probability* | *Optimal of Action* | *Course Profit from Optimal Course of Action (Rs in million)* | |
|---|---|---|---|---|
| | | | *Conditional Profit* | *Weighted Profit* |
| High demand | 0.35 | $S_1$ | 4 | 4×0.35 = 1.40 |
| Moderate demand | 0.65 | $S_2$ | 0 | 0×0.65 = 0 |
| Total | | | | EPPI = 1.40 |

The optimal EMV* is Rs 0.75 million corresponding to the course of action $S_1$. Then

$$EVPI = EPPI - EMV(S_1)$$

$$= 1.40 - 0.75 = \text{Rs } 0.65 \text{ million}$$

In other words, if the company could get accurate information (or forecast) of demand (high or moderate), it should consider paying up to Rs 0.65 million for such information.

The expected value of perfect information in business helps in getting an absolute upper bound on the amount that should be spent to get additional information on which to base a given decision.

(d) The opportunity loss values are shown in Table 19.13.

**Table 19.13: Conditional Opportunity Loss Table**

| *State of Nature (Demand)* | *Probability* | *Conditional Profit (Rs in million) due to Course of Action* | | *Conditional Opportunity Loss (Rs in million) due to Course of Action* | |
|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_1$ | $S_2$ |
| High demand ($N_1$) | 0.35 | 4 | 1 | 0 | 3 |
| Moderate demand ($N_2$) | 0.65 | – 1 | 0 | 1 | 0 |

The conditional opportunity loss values may be explained as follows: If high demand ($N_1$) occurred, then the maximum profit of Rs 4 million would be achieved by selecting course of action $S_1$. Therefore, the selection of $S_1$ would result in zero opportunity loss, as it is the best decision that can be made if $N_1$ occurs. If course of action $S_2$ was chosen with a payoff of Rs 1 million, then this would result in an opportunity loss of 4 – 1 = Rs 3 million. If moderate demand ($N_2$) occurred, then the best course of action would be $S_2$ with nil profit. Thus, opportunity loss would be associated with the selection of $S_2$ but if $S_1$ was selected, then the opportunity loss would be 0 – (– 1) = Rs 1 million. That is, the company would have been Rs 1 million worse off if it had chosen course of action $S_2$.

Using the given estimates of probabilities associated with each state of nature, that is, $P(N_1) = 0.35$, and $P(N_2) = 0.65$, the expected opportunity losses for the two courses of action are:

$$EOL(S_1) = 0.35(0) + 0.65(1) = \text{Rs } 0.65 \text{ million}$$

$$EOL(S_2) = 0.35(3) + 0.65(0) = \text{Rs } 1.05 \text{ million}$$

Since the decision-maker seeks to minimize the expected opportunity loss, he must select the course of action $S_1$ as it produces the smallest expected opportunity loss.

**Example 19.10:** A certain piece of equipment has to be purchased for a construction project at a remote location. This equipment contains an expensive part which is subject to random failure. Spares of this part can be purchased at the same time the equipment is purchased. Their unit cost is Rs 1500 and they have no scrap value. If the part fails on the job and no spare is available, the part will have to be manufactured on a special order basis. If this is required, the total cost including down time of the equipment, is estimated at Rs 9000 for each such occurrence. Based on previous experience with similar parts, the following probability estimates of the number of failures expected over the duration of the project are provided as given below:

| | | | |
|---|---|---|---|
| Failure : | 0 | 1 | 2 |
| Probability : | 0.80 | 0.15 | 0.05 |

(a) Determine the optimal EMV* and the optimal number of spares to be purchased initially.

(b) Based on opportunity losses, determine the optimal course of action and optimal value of EOL.

(c) Determine the expected profit with perfect information and expected value of perfect information.

**Solution:** (a) Let $N_1$ (no failure), $N_2$ (one failure) and $N_3$ (two failures) be the possible states of nature (i.e., number of parts failures or number of spares required). Similarly, let $S_1$ (no spare purchased), $S_2$ (one spare purchased), and $S_3$ (two spares purchased) be the possible courses of action or strategies.

The conditional costs for each pair of course of action and state of nature combination is shown in Table 19.14.

**Table 19.14**

| *State of Nature (Spares required)* | *Course of Action (Number of spares purchased)* | *Purchase Cost (Rs)* | *Emergency Cost (Rs)* | *Total Conditional Cost (Rs)* |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1500 | 0 | 1500 |
| 0 | 2 | 3000 | 0 | 3000 |
| 1 | 0 | 0 | 9000 | 9000 |
| 1 | 1 | 1500 | 0 | 1500 |
| 1 | 2 | 3000 | 0 | 3000 |
| 2 | 0 | 0 | 18,000 | 18,000 |
| 2 | 1 | 1500 | 9000 | 10,500 |
| 2 | 2 | 3000 | 0 | 3000 |

Using the conditional costs as given in Table 19.14 and the probabilities of the states of nature, the expected monetary value can be calculated for each of the three states of nature as shown in Table 19.15.

**Table 19.15: Expected Monetary Value**

| *State of Nature* | *Probability* | *Conditional cost (Rs) due to Course of Action* | | | *Weighted Cost (Rs) due to Course of Action* | | |
|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ |
| $N_1$ | 0.80 | 0 | 1500 | 3000 | 0.80(0) = 0 | 1200 | 2400 |
| $N_2$ | 0.15 | 9000 | 1500 | 3000 | 0.15(9000) = 1350 | 225 | 450 |
| $N_3$ | 0.05 | 18,000 | 10,500 | 3000 | 0.05(18,000) = 900 | 525 | 150 |
| | | | | | EMV = 2250 | 1950 | 3000 |

Since the weighted cost—Rs 1950—is lowest due to course of action, $S_2$, it should be chosen. If the EMV is expressed in terms of profit, then

$$\text{EMV}^* = \text{EMV}(S_2) = -\text{ Rs } 1950$$

Hence, the optimal number of spares to be purchased initially should be one.

(b) To determine the EOL, we must first find the COL. The calculations for conditional opportunity loss (COL) are shown in Table 19.16.

**Table 19.16: Conditional Opportunity Loss (COL)**

| *State of Nature* | *Conditional Cost (Rs) due to Course of Action* | | | *Conditional Opportunity Loss (Rs) due to Course of Action* | | |
|---|---|---|---|---|---|---|
| | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ |
| $N_1$ | 0 | 1500 | 3000 | 0 | 1500 | 3000 |
| $N_2$ | 9000 | 1500 | 3000 | 7500 | 0 | 1500 |
| $N_3$ | 18,000 | 10,500 | 3000 | 15,000 | 7500 | 0 |

Since we are dealing with conditional costs rather than conditional profits, the lower value for each state of nature shall be considered for calculating opportunity losses.

The calculations for expected opportunity loss are shown in Table 19.17.

**Table 19.17: Expected Opportunity Loss (EOL)**

| *State of Nature* | *Probability* | *Conditional Cost (Rs) due to Course of Action* | | | *Weighted Cost (Rs) due to Course of Action* | | |
|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | $S_1$ | $S_2$ | $S_3$ |
| $N_1$ | 0.80 | 0 | 1500 | 3000 | 0.80(0) = 0 | 1200 | 2400 |
| $N_2$ | 0.15 | 7500 | 0 | 1500 | 0.15 (7500) = 1125 | 0 | 225 |
| $N_3$ | 0.05 | 15,000 | 7500 | 0 | 0.05 (15,000) = 750 | 375 | 0 |
| | | | | | EMV = 1875 | 1575 | 2625 |

Since $\text{EOL}^* = \text{EOL}(S_2) =$ Rs 1,575, therefore, adopt course of action $S_2$ and purchase one spare.

(c) The expected profit with perfect information (EPPI) can be determined by selecting the optimal course of action for each state of nature, multiplying its conditional values by the corresponding probability and then summing these products. The EPPI calculations are shown in Table 19.18.

**Table 19.18**

| *States of Nature* | *Probability* | *Optimal Course of Action* | *Cost of Optimal Course of Action (Rs)* | |
|---|---|---|---|---|
| | | | *Conditional Cost* | *Weighted Opportunity Loss* |
| $N_1$ | 0.80 | $S_1$ | 0 | 0.80(0) = 0 |
| $N_2$ | 0.15 | $S_2$ | 1500 | 0.15 (1500) = 225 |
| $N_2$ | 0.05 | $S_3$ | 3000 | 0.05 (3000) = 150 |
| | | | | 375 |

Since expected profit with perfect information is Rs 375, therefore the expected value of perfect information is given by

$$\text{EVPI} = \text{EPPI} - \text{EMV}^* = -375 - (-1950) = \text{Rs } 1575$$

Here it can be observed that, EVPI = EOL* = Rs 1575.

**Example 19.11:** XYZ Company manufactures parts for passenger cars and sells them in lots of 10,000 parts each. The company has a policy of inspecting each lot before it is actually shipped to the retailer. Five inspection categories, established for quality control, represent the percentage of defective items contained in each lot. These are given in the following table. The daily inspection chart for the past 100 inspections shows the following rating or breakdown inspection.

The management is considering two possible courses of action:

(i) $S_1$: Shut down the entire plant operations and thoroughly inspect each machine.

| *Rating* | *Proportion of Defective Items* | *Frequency* |
|---|---|---|
| Excellent (A) | 0.02 | 25 |
| Good (B) | 0.05 | 30 |
| Acceptable (C) | 0.10 | 20 |
| Fair (D) | 0.15 | 20 |
| Poor (E) | 0.20 | 5 |
| | | 100 |

(ii) $S_2$: Continue production as it now exists but offer the customer a refund for defective items that are found and subsequently returned.

The first alternative will cost Rs 600 while the second alternative will cost the company Re 1 for each defective item that is returned. What is the optimum decision for the company? Find the EVPI.

**Solution:** Calculations of inspection and refund cost are shown in Table 19.19.

**Table 19.19: Inspection and Refund Cost**

| *Rating* | *Defect Rate* | *Probability* | *Cost* | | *Opportunity Loss* | |
|---|---|---|---|---|---|---|
| | | | *Inspect* | *Refund* | *Inspect* | *Refund* |
| A | 0.02 | 0.25 | 600 | 200 | 400 | 0 |
| B | 0.05 | 0.30 | 600 | 500 | 100 | 0 |
| C | 0.10 | 0.20 | 600 | 1000 | 0 | 400 |
| D | 0.15 | 0.20 | 600 | 1500 | 0 | 900 |
| E | 0.20 | 0.05 | 600 | 2000 | 0 | 1400 |
| | | 1.00 | 600* | 670 | 170* | 240 |

The cost of refund is calculated as follows:

$$\text{For lot A: } 10{,}000 \times 0.02 \times 1.00 = \text{Rs } 200$$

The cost of refund for other lots is calculated in a similar manner.

The expected cost of refund is:

$$200 \times 0.25 + 500 \times 0.30 + \ldots + 2000 \times 0.05 = \text{Rs } 670$$

Now, the expected cost of inspection is:

$$600 \times 0.25 + 600 \times 0.30 + \ldots + 600 \times 0.05 = \text{Rs } 600$$

Since the cost of refund is more than the cost of inspection, the plant should be shut down for inspection. Also, EVPI = EOL of inspection = Rs 170

**Example 19.12:** A toy manufacturer is considering a project for manufacturing a dancing doll with three different movement designs. The doll will be sold at an average price of Rs 10. The first movement design using 'gears and levels' will provide the lowest tooling at a set up cost of Rs 1,00,000 and Rs 5 per unit of variable cost. A second design with spring action will have a fixed cost of Rs 1,60,000 and variable cost of Rs 4 per unit. Yet another design with weights and pulleys will have a fixed cost of Rs 3,00,000 and variable cost Rs 3 per unit. One of the following demand events and its probabilities can occur for the doll:

| | *Demand (units)* | *Probability* |
|---|---|---|
| Light demand | 25,000 | 0.10 |
| Moderate demand | 1,00,000 | 0.70 |
| Heavy demand | 1,50,000 | 0.20 |

(a) Construct a payoff table for the above project.

(b) Which is the optimum design?

(c) How much can the decision-maker afford to pay to obtain perfect information about the demand?

**Solution:** The calculations for EMV are shown in Table 19.20.

Payoff = (Demand × Selling price) – (Fixed cost + Demand × Variable cost)
= Revenue – Total variable cost – Fixed cost

**Table 19.20: EMV and Payoff Values**

| *States of Nature (Demand)* | *Probability* | *Conditional Payoff (Rs) due to Courses of Action (Choice of Movements)* | | | *Expected Payoff (Rs) due to Courses of Action* | | |
|---|---|---|---|---|---|---|---|
| | | *Gears and Levels* | *Spring Action* | *Weights and Pulleys* | *Gears and Levels* | *Spring Action* | *Weights and Pulleys* |
| Light | 0.10 | 25,000 | – 10,000 | – 1,25,000 | 2500 | – 1000 | – 12,500 |
| Moderate | 0.70 | 4,00,000 | 4,40,000 | 4,00,000 | 2,80,000 | 3,08,000 | 2,80,000 |
| Heavy | 0.20 | 6,50,000 | 7,40,000 | 7,50,000 | 1,30,000 | 1,48,000 | 1,50,000 |
| | | | | EMV | 4,12,500 | 4,55,000 | 4,17,500 |

Since EMV is largest for spring action, it must be selected.

**Table 19.21: Expected Payoff with Perfect Information**

| *States of Nature (Demand)* | *Probability* | *Courses of Action* | | | *Maximum Payoff* | *Maximum Payoff × Probability* |
|---|---|---|---|---|---|---|
| | | *Gears and Levels* | *Spring Action* | *Weights and Pulleys* | | |
| Light | 0.10 | 25,000 | – 10,000 | – 1,25,000 | 25,000 | 2500 |
| Moderate | 0.70 | 4,00,000 | 4,40,000 | 4,00,000 | 4,40,000 | 3,08,000 |
| Heavy | 0.20 | 6,50,000 | 7,40,000 | 7,50,000 | 7,50,000 | 1,50,000 |
| | | | | | | 4,60,500 |

The maximum amount of money that the decision-maker would be willing to pay to obtain perfect information regarding demand for the doll will be

EVPI = Expected payoff with perfect information – Expected payoff under uncertainty (EMV)
= 4,60,500 – 4,55,000 = Rs 5500

**Example 19.13:** The demand pattern of cakes made in a bakery is as follows:

| No. of cakes demanded : | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| Probability : | 0.05 | 0.10 | 0.25 | 0.30 | 0.20 | 0.10 |

If the preparation cost is Rs 3 per unit and selling price is Rs 4 per unit, how many cakes should the baker bake to maximize profit?

**Solution:** Given that incremental cost (IC) is Rs 3 per unit and incremental price (IP) Rs 4 per unit, the minimum required probability of selling at least an additional unit of cake to justify the stocking of that unit is given by

$$p = \frac{IC}{IC + IP} = \frac{3}{3+4} = 0.75$$

This probability means that the bakery owner must have demand level $k$ such that P (demand $\geq k$) $\geq p$.

The cumulative probabilities of greater than type are computed as shown in Table 19.22.

**Table 19.22**

| *Demand (No. of cakes)* | *Probability P (Demand = k)* | *Cumulative Probability P (Demand ≥ k)* |
|---|---|---|
| 0 | 0.05 | 1.00 |
| 1 | 0.10 | 0.95 |
| 2 | 0.25 | 0.85 ← |
| 3 | 0.30 | 0.60 |
| 4 | 0.20 | 0.30 |
| 5 | 0.10 | 0.10 |

Since the highest value of $k$ for which P (demand $\geq k$) exceeds the critical ratio $p$ = 0.75 is $k$ = 2, the optimal decision is to prepare only two cakes.

## 19.6 POSTERIOR PROBABILITIES AND BAYESIAN ANALYSIS

**Posterior analysis:** A procedure for determining the optimal decision based on states of nature resulting from combining the prior probability distribution with information obtained from an experimentation.

The search and evaluation of decision alternatives often reveal new information. If such information is regarding the identification of alternatives, it requires revision and expansion of the test of alternatives. But if it is regarding the effects of alternatives, consequences are restated. When uncontrollable factors are involved, either the states of nature themselves are reconsidered or their likelihoods are revised. The value of new information is evaluated in terms of its impact on the expected payoff. The expected value and the cost of the new information are compared to determine whether it is worth acquiring.

An initial probability statement to evaluate expected payoff is called a *prior probability distribution*. The one which has been revised in the light of new information is called a *posterior probability distribution*. It will be evident that what is a posterior to one sequence of state of nature becomes the prior to others which are yet to happen.

This section will be concerned with the method of computing posterior probabilities from prior probabilities using a mathematical formula called *Baye's theorem*. A further analysis of problems using these probabilities with respect to new expected payoffs with additional information is called *prior-posterior analysis*. The Baye's theorem, in general terms, can be stated as follows:

Let $A_1, A_2, \ldots, A_n$ be mutually exclusive and collectively exhaustive outcomes. Their probabilities $P(A_1), P(A_2), \ldots, P(A_n)$ are known. There is an experimental outcome B for which the conditional probabilities $P(B \mid A_1), P(B \mid A_2), \ldots, P(B \mid A_n)$ are also known.

Given the information that outcome B has occurred, the revised conditional probabilities of outcomes $A_i$, that is, $P(A_i \mid B)$, $i = 1, 2, \ldots, n$ are determined by using following conditional probability relationship:

$$P(A_i \mid B) = \frac{P(A_i \text{ and } B)}{P(B)} = \frac{P(A_i \cap B)}{P(B)}$$

where $$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$

Since each joint probability can be expressed as the product of a known marginal (prior) and conditional probability,

$$P(A_i \cap B) = P(A_i) \times P(B \mid A_i)$$

Thus $$P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{P(A_1)\, P(B \mid A_1) + P(A_2)\, P(B \mid A_2) + \ldots + P(A_n)\, P(B \mid A_n)}$$

**Example 19.14:** A company is considering the introduction of a new product to its existing product range. It has defined two levels of sales as 'high' and 'low' on which to base its decision and has estimated the changes that each market level will occur, together with their costs and consequent profits or losses. The information is summarized below:

| *States of Nature* | *Probability* | *Courses of Action* | |
|---|---|---|---|
| | | *Market the Product (Rs in '000)* | *Do not Market the Product (Rs in '000)* |
| High sales | 0.3 | 150 | 0 |
| Low sales | 0.7 | −40 | 0 |

The company's marketing manager suggests that a market research survey may be undertaken to provide further information on which to base the decision. On past experience with a certain market research organization, the marketing manager assesses its ability to give good information in the light of subsequent actual sales achievements as follows:

| *Market Research (Survey outcome)* | *Actual Sales* | |
|---|---|---|
| | *Market 'high'* | *Market 'low'* |
| 'High' sales forecast0.5 | 0.1 | |
| Indecisive survey report | 0.3 | 0.4 |
| 'Low' sales forecast0.2 | 0.1 | |

The market research survey will cost Rs 20,000, state whether or not there is a case for employing the market research organization. *[Delhi Univ., MBA, 1996]*

**Solution:** The expected monetary value (EMV) for each course of action is shown in Table 19.23.

**Table 19.23**

| *States of Nature* | *Probability* | *Courses of Action* | | *Expected Profit (Rs in '000)* | |
|---|---|---|---|---|---|
| | | *Market* | *Do not Market* | *Market* | *Do not Market* |
| High sales | 0.3 | 150 | 0 | 45 | 0 |
| Low sales | 0.7 | −40 | 0 | −28 | 0 |
| | | | | EMV = 17 | = 0 |

With no additional information, the company should choose course of action 'market the product'. However, if the company had the perfect information about the 'low sales' then it would not go ahead as the expected value is – Rs 28,000. Thus, the value of perfect information is the expected value of low sales.

Let us define the outcomes of the research survey as: high sales ($S_1$), indecisive report ($S_2$), and low sales ($S_3$), and states of nature as: high market ($N_1$) and low market ($N_2$)

The calculations for prior probabilities of forecast are given in Table 19.24.

**Table 19.24**

| *Outcome* | *Sales Prediction* | |
|---|---|---|
| | *High Market ($N_1$)* | *Low Market ($N_2$)* |
| High sales ($S_1$) | $P(S_1 \mid N_1) = 0.5$ | $P(S_1 \mid N_2) = 0.1$ |
| Indecisive report ($S_2$) | $P(S_2 \mid N_1) = 0.3$ | $P(S_2 \mid N_2) = 0.4$ |
| Low sales ($S_3$) | $P(S_3 \mid N_1) = 0.2$ | $P(S_3 \mid N_2) = 0.5$ |

With this additional information, the company can now revise the prior probabilities of outcomes to get posterior probabilities. These can be used to recalculate the EMV and determine the optimal course of action given the additional information. Table 19.25 shows the calculation of the revised probabilities given the sales forecast.

**Table 19.25: Revised Probabilities**

| *States of Nature* | *Prior Probability $P(N_i)$* | *Conditional Probability $P(S_i \mid N_i)$* | *Joint Probability $P(S_i \cap N_i) = P(N_i)\,P(S_i \mid N_i)$* | | |
|---|---|---|---|---|---|
| High sales ($N_1$) | 0.3 | $P(S_1 \mid N_1) = 0.5$ | 0.15 | — | — |
| | | $P(S_2 \mid N_1) = 0.3$ | — | 0.09 | — |
| | | $P(S_3 \mid N_1) = 0.2$ | — | — | 0.06 |
| Low sales ($N_2$) | 0.7 | $P(S_1 \mid N_2) = 0.1$ | 0.07 | — | — |
| | | $P(S_2 \mid N_2) = 0.4$ | — | 0.28 | — |
| | | $P(S_3 \mid N_2) = 0.5$ | — | — | 0.35 |
| Marginal Probability | | | 0.22 | 0.37 | 0.41 |

The posterior probabilities of actual sales given the sales forecast are:

| *Outcome ($S_i$)* | *Probability $P(S_i)$* | *States of Nature ($N_i$)* | *Posterior Probability $P(N_i \mid S_i) = P(N_i \cap S_i) / P(S_i)$* |
|---|---|---|---|
| $S_1$ | 0.22 | $N_1$ | 0.15/0.22 = 0.681 |
| | | $N_2$ | 0.07/0.22 = 0.318 |
| $S_2$ | 0.37 | $N_1$ | 0.09/0.37 = 0.243 |
| | | $N_2$ | 0.28/0.37 = 0.756 |
| $S_3$ | 0.41 | $N_1$ | 0.06/0.41 = 0.146 |
| | | $N_2$ | 0.35/0.41 = 0.853 |

For each outcome, the revised probabilities are now used to calculate the net expected value (EV) given the additional information supplied by that outcome as shown in Table 19.26.

**Table 19.26**

| *States of Nature* | *Revised Conditional Profit (Rs)* | *Sales Forecast* | | | | | |
|---|---|---|---|---|---|---|---|
| | | *High* | | *Indecisive* | | *Low* | |
| | | *Prob.* | *EV (Rs)* | *Prob.* | *EV (Rs)* | *Prob.* | *EV (Rs)* |
| High sales → | 130 | 0.681 | 88.53 | 0.243 | 31.59 | 0.146 | 18.98 |
| Low sales → | – 60 | 0.318 | –19.08 | 0.756 | – 45.36 | 0.853 | –51.18 |
| Expected value of sales forecast | | | 69.45 | | – 13.77 | | – 32.20 |
| Probability of occurrence | | | 0.22 | | 0.37 | | 0.41 |
| Net expected value (Expected value × Probability) | | | 15.279 | | – 5.095 | | 13.202 |

**Example 19.15:** A farmer is attempting to decide which of three crops he should plant on his 100 acre farm. The profit from each crop is strongly dependent on the rainfall during the growing season. He has categorized the amount of the rainfall as substantial, moderate, or light. He estimates his profit for each crop to be as shown in the table below:

| *Rainfall* | *Estimated Profit (Rs)* | | |
|---|---|---|---|
| | *Crop A* | *Crop B* | *Crop C* |
| Substantial | 7000 | 2500 | 4000 |
| Moderate | 3500 | 3500 | 4000 |
| Light | 1000 | 4000 | 3000 |

Based on the weather in previous seasons and the current projection for the coming season, he estimates the probability of substantial rainfall as 0.2, that of moderate rainfall as 0.3, and that of light rainfall as 0.5.

Furthermore, the services of forecasters could be employed to provide a detailed survey of current rainfall prospects as shown in the table.

| *Rainfall* | *Estimated Profit (Rs)* | | |
|---|---|---|---|
| | *Crop A* | *Crop B* | *Crop C* |
| Substantial | 0.70 | 0.25 | 0.05 |
| Moderate | 0.30 | 0.60 | 0.10 |
| Light | 0.10 | 0.20 | 0.70 |

(a) From the available data, determine the optimal decision as to which crop to plant.
(b) Determine whether it would be economical for the farmer to hire the services of a forecaster.

**Solution:** (a) Let $N_i$ be the state of nature ($i$ = 1, 2, 3) representing 'substantial rainfall', 'moderate rainfall', and 'light rainfall' respectively, and $S_j$ be the course of action ($j$ = 1, 2, 3) representing 'Crop A', 'Crop B', and 'Crop C', respectively.

**Table 19.27: Calculation of EMVs**

| *States of Nature* | *Prior Probability* | *Conditional Profit (Rs) Courses of Action* | | | | *Expected Profit (Rs) Courses of Action* | | |
|---|---|---|---|---|---|---|---|---|
| | | $S_1$ | $S_2$ | $S_3$ | | $S_1$ | $S_2$ | $S_3$ |
| $N_1$ | 0.2 | **7000** | 2500 | 4000 | | 1400 | 500 | 800 |
| $N_2$ | 0.3 | 3500 | 3500 | **4000** | | 1050 | 1050 | 1200 |
| $N_3$ | 0.5 | 1000 | **4000** | 3000 | | 500 | 2000 | 1500 |
| | | | | | EMV = | 2950 | 3550 | 3500 |

The maximum EMV is Rs 3550. Therefore, the optimal course of action is $N_2$, that is, plant crop B. However, it would make no sense to plant more than one kind of crop because maximum EMV is obtained by planting all 100 acres with crop B.

(b) Let $B_i$ ($i$ = 1, 2, 3) denote the outcome forecast for 'substantial rainfall', 'moderate rainfall', and 'light rainfall' respectively. The likelihood values are given in Table 19.28.

**Table 19.28**

| *States of Nature* ($N_i$) | *Forecast Likelihood* $P(B_1 \mid N_i)$ | $P(B_2 \mid N_i)$ | $P(B_3 \mid N_i)$ |
|---|---|---|---|
| $N_1$ | 0.70 | 0.25 | 0.05 |
| $N_2$ | 0.30 | 0.60 | 0.10 |
| $N_3$ | 0.10 | 0.20 | 0.70 |

From the data of Table 19.27, where maximum profit for each state of nature is written in bold, the expected profit with perfect information is given by

$$\text{EPPI} = 0.2(7000) + 0.3(4000) + 0.5(4000) = \text{Rs } 4600$$

Thus, we have $\text{EVPI} = \text{EPPI} - \text{EMV}^* = 4600 - 3550 = \text{Rs } 1050$

For each of the three forecast results, the prior and posterior probabilities are given in Tables 19.29 and 19.30.

**Table 19.29**

| *States of Nature* | *Prior Probability* | *Outcomes* ($B_i$) | *Conditional Probability* $P(B_i \mid N_i)$ | *Joint Probability* $P(B_i \cap N_i) = P(N_i)\,P(B_i \mid N_i)$ | | |
|---|---|---|---|---|---|---|
| $N_1$ | 0.2 | $B_1$ | 0.70 | 0.14 | — | — |
| | | $B_2$ | 0.25 | — | 0.05 | — |
| | | $B_3$ | 0.05 | — | — | 0.01 |
| $N_2$ | 0.3 | $B_1$ | 0.30 | 0.09 | — | — |
| | | $B_2$ | 0.60 | — | 0.18 | — |
| | | $B_3$ | 0.10 | — | — | 0.03 |
| $N_3$ | 0.5 | $B_1$ | 0.10 | 0.05 | — | — |
| | | $B_2$ | 0.20 | — | 0.10 | — |
| | | $B_3$ | 0.70 | — | — | 0.35 |
| Marginal probability | | | | 0.28 | 0.33 | 0.39 |

**Table 19.30**

| *Outcome* ($B_i$) | *Probability* $P(B_i)$ | *States of Nature* ($N_i$) | *Posterior Probability* $P(N_i \mid B_i) = P(N_i \cap B_i)/P(B_i)$ |
|---|---|---|---|
| $B_1$ | 0.28 | $N_1$ | 0.14/0.28 = 0.500 |
| | | $N_2$ | 0.09/0.28 = 0.321 |
| | | $N_3$ | 0.05/0.28 = 0.178 |
| $B_2$ | 0.33 | $N_1$ | 0.05/0.33 = 0.151 |
| | | $N_2$ | 0.18/0.33 = 0.303 |
| | | $N_3$ | 0.10/0.33 = 0.030 |
| $B_3$ | 0.39 | $N_1$ | 0.01/0.39 = 0.025 |
| | | $N_2$ | 0.03/0.39 = 0.076 |
| | | $N_3$ | 0.35/0.39 = 0.897 |

For each outcome, the revised probabilities are now used to recalculate the EMVs, given the additional information supplied by that outcome, as shown in Table 19.31.

**Table 19.31**

| *States of Nature ($N_i$)* | *Forecast Outcome* | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $B_1$ | | | $B_2$ | | | $B_3$ | | |
| | *Prob.* | *COL* | *EOL* | *Prob.* | *COL* | *EOL* | *Prob.* | *COL* | *EOL* |
| $N_1$ | 0.500 | 0 | 0 | 0.151 | 500 | 75 | 0.025 | 3000 | 60 |
| $N_2$ | 0.321 | 4500 | 1440 | 0.303 | 500 | 275 | 0.076 | 0 | 0 |
| $N_3$ | 0.178 | 3000 | 540 | 0.030 | 0 | 0 | 0.897 | 1000 | 900 |
| Posterior | EOL | | 1980 | | | 350 | | | 960 |

The expected value of sample information can be obtained by multiplying posterior EOLs with the revised probabilities as shown in Table 19.32.

**Table 19.32**

| *Outcome ($B_i$)* | *Probability $P(B_i)$* | *Expected Opportunity Loss (EOL)* | *Expected Value of Sample Information (EVSI)* |
|---|---|---|---|
| $B_1$ | 0.28 | 1980 | 554.4 |
| $B_2$ | 0.33 | 350 | 115.5 |
| $B_3$ | 0.39 | 960 | 374.4 |
| | | | 1044.3 |

The EVSI Rs 1044.3 indicates the money which the farmer has to pay for hiring the services of a forecaster.

## Self-Practice Problems 19B

**19.6** You are given the following payoffs of three acts $A_1$, $A_2$, and $A_3$ and the events $E_1$, $E_2$, $E_3$.

| *States of Nature* | *Three Acts* | | |
|---|---|---|---|
| | $A_1$ | $A_3$ | $A_3$ |
| $E_1$ | 25 | – 10 | – 125 |
| $E_2$ | 400 | 440 | 400 |
| $E_3$ | 650 | 740 | 750 |

The probabilities of the states of nature are 0.1, 0.7, and 0.2 respectively. Calculate and tabulate, EMV and conclude which of the course of action can be chosen as the best.

**19.7** A management is faced with the problem of choosing one of three products for manufacturing. The potential demand for each product may turn out to be good, moderate, or poor. The probabilities for each of the states of nature were estimated as follows:

| *Product* | *Nature of Demand* | | |
|---|---|---|---|
| | *Good* | *Moderate* | *Poor* |
| X | 0.70 | 0.20 | 0.10 |
| Y | 0.50 | 0.30 | 0.20 |
| Z | 0.40 | 0.50 | 0.10 |

The estimated profit or loss in rupees under the three states may be taken as:

| *Product* | *Good* | *Moderate* | *Poor* |
|---|---|---|---|
| X | 30,000 | 20,000 | 10,000 |
| Y | 60,000 | 30,000 | 20,000 |
| Z | 40,000 | 10,000 | – 15,000 |

Prepare the expected value table, and advise the management about the choice of product. [*CA, May 1986*]

**19.8** The marketing staff of a certain industrial organization has submitted the following payoff table, giving profits in million rupees, concerning a certain proposal depending upon the rate of technology advance.

| *Technological Advance* | *Decision* | |
|---|---|---|
| | *Accept* | *Reject* |
| Much | 2 | 3 |
| Little | 5 | 2 |
| None | (–)1 | 4 |

The probabilities are 0.2, 0.5, and 0.3 for Much, Little, and None technological advance respectively. What decision should be taken?

**19.9** A physician purchases a particular vaccine on Monday of each week. The vaccine must be used within the following week, otherwise it becomes worthless. The vaccine costs Rs 2 per dose and the physician charges Rs 4 per dose. In the past 50 weeks, the physician has administered the vaccine in the following quantities:

| | | | | |
|---|---|---|---|---|
| Doses per week : | 20 | 25 | 40 | 60 |
| Number of weeks : | 5 | 15 | 25 | 5 |

Determine how many doses the physician should buy every week.

**19.10** A grocery with a bakery department is faced with the problem of how many cakes to buy in order to meet the day's demand. The grocer prefers not to sell day-old goods in competition with fresh products; leftover cakes are, therefore, a complete loss. On the other hand, if a customer desires a cake and all of them have been sold, the disappointed customer will buy from elsewhere and the sale will be lost. The grocer has, therefore, collected information on the past sales on a selected 100-day period as shown in table below:

| *Sales per Day* | *No. of Days* | *Probability* |
|---|---|---|
| 25 | 10 | 0.10 |
| 26 | 30 | 0.30 |
| 27 | 50 | 0.50 |
| 28 | 10 | 0.10 |

Construct the payoff table and the opportunity loss table. What is the optimal number of cakes that should be bought each day? Also find and interpret the EVPI (Expected Value of Perfect Information). A cake costs Re 0.80 and sells for Re 1.

## Hints and Answers

**19.6** EMV($A_1$) = 412.5, EMV($A_2$) = 455, EMV($A_3$) = 417.5

**19.7** EMV(X) = 26, EMV(Y) = 43, EMV(Z) = 19.5; Company should manufacture product Y.

**19.8** EMV (accept) = 2.6, EMV (reject) = 2.8; reject.

**19.9** Conditional profit value
= MP × units sold – ML × units unsold

$$= \begin{cases} (4-2)D = 2D & ;\ D \geq S \\ (4-2)D - 2(S-D) = 4D - 2S & ;\ D < S \end{cases}$$

where D is the number of units demanded and S is the number of units stocked

EMV* = EMV (Purchase 40 dozen) = Rs 54.

**19.10** Conditional profit value
= MP × cake sold – ML × cake not sold
= (1 – 0.80) × cake sold – 0.80 × cake not sold

$$= \begin{cases} 0.20D & ;\ D \geq S \\ 0.20D - 0.80\,(S-D) & ;\ D < S \end{cases}$$

where D is the number of units demanded and S is the number of units stocked

EMV* = Rs 5 ; EOL* = 0.22 (stock 26 units of cake)

## 19.7 DECISION TREE ANALYSIS

The decision-making problems discussed so far are referred to as single stage decision problems, because the payoffs, states of nature, courses of action, and probabilities associated with the occurrence of states of nature are not subject to change.

However, situations may arise when a decision-maker needs to revise his previous decisions on getting new information and make a sequence of other decisions. Thus, the problem becomes a multi-stage decision problem because the consequence of one decision affects future decisions. For example, in the process of marketing a new product, the first decision is often test marketing and the alternative courses of action might be either intensive testing or gradual testing. Given various possible consequences—good, fair, or poor, the decision-maker may be required to decide between redesigning the product, an aggressive advertising campaign, or complete withdrawal of product, and so on. Given that decision, there will be an outcome which will lead to another decision and so on.

Decision tree analysis involves the construction of a diagram showing all the possible courses of action, states of nature, and the probabilities associated with the states of nature. The *decision diagram* looks very much like a drawing of a tree, therefore also called **decision-tree.**

**Decision tree:** A graphical presentation for displaying acts and events in a decision problem in the form of a tree diagram.

A decision tree consists of *nodes, branches, probability estimates,* and *payoffs*. There are two types of nodes: *decision nodes* and *chance nodes*. A decision node is usually represented by a square and indicates places where a decision-maker must make a decision. Each branch leading away from a decision node represents one of the several possible courses of action available to the decision-maker. The chance node is represented by a circle and indicates a point at which the decision-maker will discover the response to his decision, that is, different possible outcomes occurring from a chosen course of action.

*Branches* emanate from and connect various nodes and are either decisions or states of nature. There are two types of branches: *decision branches* and *chance branches*. Each branch leading away from a decision node represents a course of action or strategy that can be chosen at a decision point, whereas a branch leading away from a chance node represents the state of nature of a set of chance factors. Associated probabilities are indicated alongside of the respective chance branch. These probabilities are the likelihood that the chance outcome will assume the value assigned to the particular branch. Any branch that makes the end of the decision tree, that is, it is not followed by either another decision or chance node, is called a *terminal branch*. A terminal branch can represent either a course of action or a chance outcome. The terminal points of a decision tree are said to be mutually exclusive points so that exactly one course of action will be chosen.

The *payoff* can be positive (i.e., revenue or sales) or negative (i.e., expenditure or cost) and they can be associated either with decisions or chance branches.

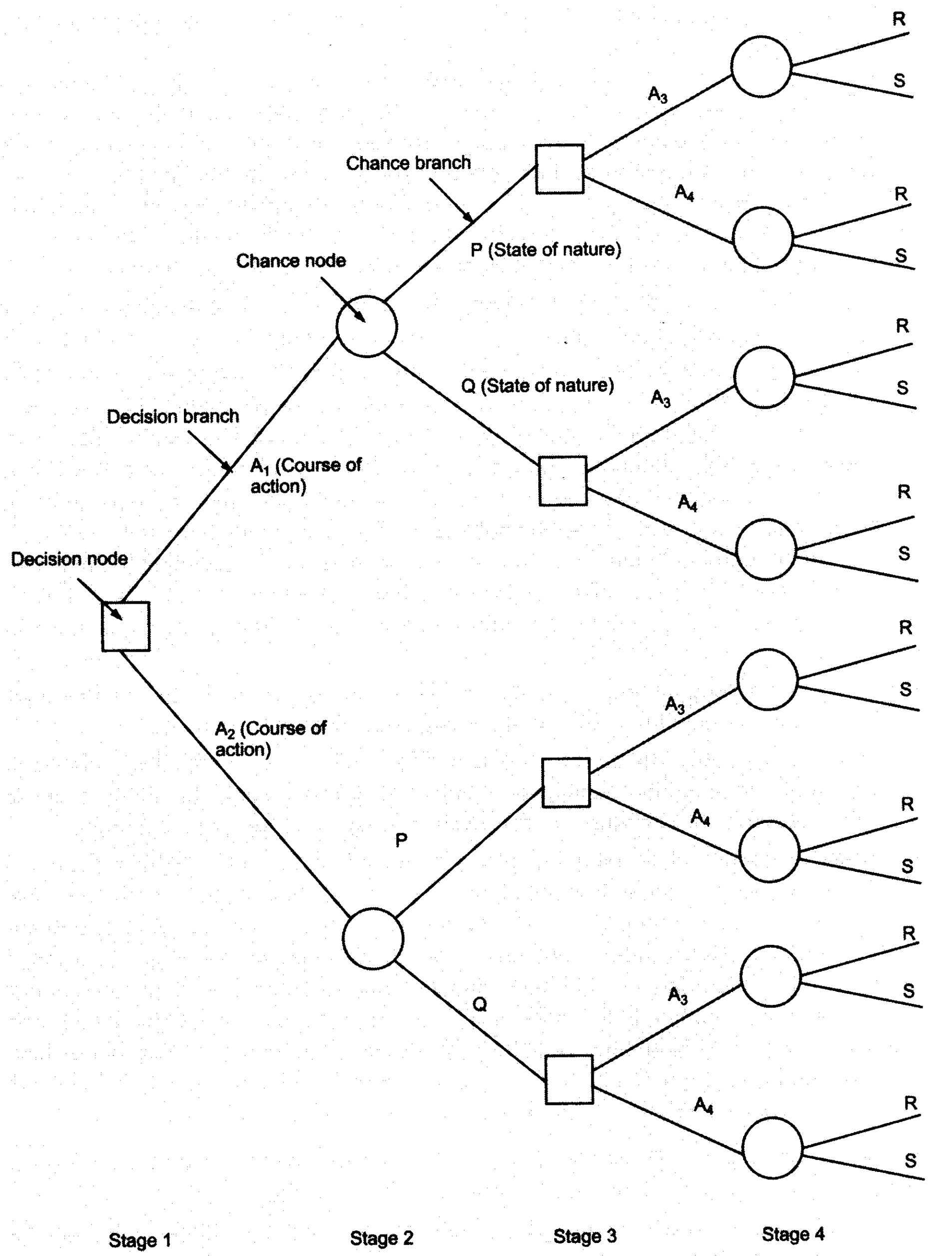
An illustration of a decision tree is shown in Fig. 19.1. It is possible for a decision tree to be deterministic or probabilistic and it can be further divided into single stage (a decision under condition of certainty) and a multistage (a sequence of decisions).

The optimal sequence of decisions in a tree is found by starting at the right-hand side and rolling backward. The aim of this operation is to maximize the return from the decision situation. At each node, an expected return is calculated (called the *position value*). If the node is a chance node, then the position value is calculated as the sum of the products of the probabilities or the branches emanating from the chance node and their respective position values. If the node is a decision node, then the expected return is calculated for each of its branches and the highest return is selected. The procedure continues until the initial node is reached. The position values for this node correspond to the maximum expected return obtainable from the decision sequence.

**Remarks:** *Decision Trees Versus Probability Trees:* Decision trees are basically an extension of probability trees. However, there are several basic differences:

1. The decision tree utilizes the concept of 'rollback' to solve a problem. This means starting at the right-hand terminus with the highest expected value of the tree and working back to the current or beginning decision point to determine the decision or decisions that should be made. Most decisions require trees with numerous branches and more than one decision point. It is the multiplicity of decision points that make the rollback process necessary.
2. The probability tree is primarily concerned with calculating the correct probabilities, whereas the decision tree utilizes probability factors as a means of arriving at a final answer.
3. A most important feature of the decision tree, not found in probability trees, is that it takes time differences of future earnings into account. At any stage of the decision tree, it may be necessary to weigh the differences in immediate cost or revenue against differences in value at the next stage.

**Figure 19.1**
Decision Tree




**Example 19.16:** You are given the following estimates concerning a research and development programme:

| *Decision* $D_i$ | *Probability of Decision* $D_i$ *Given Research R* $P(D_i \mid R)$ | *Outcome Number* | *Probability of Outcome* $x_i$ *Given* $D_i$ $P(x_i \mid D_i)$ | *Payoff Value of Outcome,* $x_i$ *(Rs in '000)* |
|---|---|---|---|---|
| Develop | 0.5 | 1 | 0.6 | 600 |
| | | 2 | 0.3 | – 100 |
| | | 3 | 0.1 | 0 |
| Do not develop | 0.5 | 1 | 0.0 | 600 |
| | | 2 | 0.0 | – 100 |
| | | 3 | 1.0 | 0 |

Construct and evaluate the decision tree diagram for the above data. Show your workings for evaluation.

**Solution:** The decision tree of the given problem along with necessary calculations is shown in Fig. 19.2.